Is there an algebraic geometry for exponential sums?

B. Kazarnovskii

Abstract. An analytic set in $\mathbb{C}^n$, given as the zero set of a finite system of exponential sums, is said to be the exponential variety (E-variety). We define the intersection number for any two E-varieties. The main problem in this definition is the infinity of a 0-dimensional E-variety (as for the zero set of $\mathrm{e}^z - 1$). To overcome this obstacle, we introduce the concept of weak density, which is analogous to the number of points of a 0-dimensional algebraic variety.

Introduction

An exponential sum (ES) is a function on $\mathbb{C}^n$ of the form

$$f(z) = \sum_{\lambda\in\Lambda,\ c_\lambda\in\mathbb{C}} c_\lambda\, \mathrm{e}^{\langle z,\lambda\rangle},$$

where Λ is a finite set in $\mathbb{C}^n$, and $\langle z,\lambda\rangle = z_1\overline{\lambda}_1 + \ldots + z_n\overline{\lambda}_n$. The sets Λ and $\mathrm{conv}(\Lambda)$ are respectively called the *support* and the *Newton polytope* of ES. Below we assume that $\lambda_i \in \mathbb{R}$. Thus Newton polytope is a convex polytope in $\mathbb{R}^n$.

The ring of ESs looks like a Laurent polynomial ring. In 1929 J. Ritt proved that, if the ratio of two ESs in one variable is an entire function, then this function is also an ES. (Ritt multidimensional theorem was proved later.) However, many attempts to find other algebraic-geometric properties, similar to the properties of the ring of polynomials, encountered great difficulties. For example, the existence of a common zero of two ESs does not imply the existence of a common divisor: the ESs $\mathrm{e}^z - 1,\ \mathrm{e}^{\sqrt{2}z} - 1$, having a common zero at $z = 0$ have no a common divisor. This follows from the infinity of the zero set of any ES. It is probable that Ritt himself proposed the conjecture about the finiteness of the set of common zeros of two coprime ESs in one variable. Currently, this conjecture is very far from being proven. If one of the ESs is $\mathrm{e}^z - 1$, then the conjecture is true. This is the classical result of Skolem on Diophantine solutions of exponential equations. This follows from a theorem called the "Mordell-Lang conjecture" for a complex torus.

Description of results

There has been some progress in the algebra of ESs in recent years; see [3, 4, 5, 6]. The main result is the construction of the "ring of conditions" for the space $\mathbb{C}^n$. The ring of conditions is the ring of the intersection theory on spherical varieties; see [2, 1]. It turned out that a similar intersection theory can be constructed for E-varieties in $\mathbb{C}^n$.

The construction of the ring of conditions is based on the concept of intersection indices of E-varieties. In the talk, we define the weak density of 0-dimensional E-variety, which is analogous to the number of points of a 0-dimensional algebraic variety, and then use the weak density to define the intersection numbers. It turns out that, as in polynomial case, the intersection index of n exponential hypersurfaces is equal to the mixed volume of their Newton polytopes.

Given the intersection indices, it is easy to define the ring of conditions. By definition, the elements of the ring of conditions are the numerical equivalence classes of E-varietes with the "union" and "intersect" operations. However, proving the correctness of the definition is technically quite difficult. We will not consider the details and justify the correctness of this definition during the talk.

References

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B. Kazarnovskii
Institute for Information Transmission Problems
Moscow
e-mail: kazbori@gmail.com