Is there an algebraic geometry for exponential sums?

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The ring of exponential sums

An exponential sum (ES) is a function on $\mathbb{C}^n$ of the form

$$f(z) = \sum_{\lambda\in\Lambda,\, c_\lambda\in\mathbb{C}} c_\lambda\, \mathrm{e}^{\langle z,\lambda\rangle},$$

where Λ is a finite set in $\mathbb{C}^n$, and $\langle z,\lambda\rangle = z_1\overline{\lambda}_1 + \ldots + z_n\overline{\lambda}_n$. The sets Λ and $\mathrm{conv}(\Lambda)$ are respectively called the *support* and the *Newton polyhedron* of ES. If $\lambda_i \in \mathbb{R}$, then ES is called quasi-algebraic. *Below we consider only quasi-algebraic* ES*s.* Thus Newton polyhedra are the convex polyhedra in $\mathbb{R}^n$. An analytic set in $\mathbb{C}^n$, given as the zero set of a finite system of exponential sums, is said to be the *exponential variety (E-variety)*.

The ring ESs seems to be similar to the ring of Laurent polynomials. At the beginning of the 20th century, there were hopes to find its algebraic properties analogous to those of a polynomial ring and construct the algebraic geometry of E-varieties.

History, I

In 1929 J. Ritt proved that, if the ratio of two ESs in one variable is an entire function, then this function is also an ES. For polynomials it follows from the algorithm of division with remainder. For ESs such an algorithm does not work. So it's not obvious a priori.

Ritt multidimensional theorem was proved later (in the 70s).

However, many attempts to find other algebraic-geometric properties, similar to the properties of the ring of polynomials, encountered great difficulties.

For example, the existence of a common zero of two ESs does not imply the existence of a common divisor: the ESs $\mathrm{e}^z - 1$, $\mathrm{e}^{\sqrt{2}z} - 1$, having a common zero at $z = 0$ have no a common divisor. This follows from the infinity of the zero set of any ES (as the zero set of $\mathrm{e}^z - 1$).

History, II

It is probable that Ritt himself proposed the conjecture about the finiteness of the set of common zeros of two coprime ESs in one variable. Despite numerous efforts this conjecture is very far from being proven.

If one of the ESs is $e^z - 1$, then the conjecture is true (This follows from a theorem called the "Mordell-Lang conjecture" for a complex torus). This is the classical result of Skolem on Diophantine solutions of exponential equations.

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Skolem theorem

The set of integer zeros of ES $\sum c_\alpha e^{i\alpha z}$ can be any finite set. If the set of zeros is infinite, then it is a union of a finite set and a finite set of arithmetic progressions.

Introduction, I

There has been some progress in the algebra of ESs in recent years. The main result is the construction of the "ring of conditions" for $\mathbb{C}^n$. The ring of conditions is the ring of the intersection theory on homogeneous spherical varieties. The similar intersection theory can be constructed for E-varieties.

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The classical Chow ring is good for compact varieties. The Chow ring of a compact toric variety is isomorphic to the ring of integer cohomologies. But it is bad for non-compact varieties. The Chow ring of $(\mathbb{C} \setminus 0)^n$ is $\mathbb{Z}$. This ring knows nothing about Newton polyhedra of algebraic varieties. On the contrary, the ring condition of a torus contains all this information.

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The construction of the ring of conditions is based on the concept of *intersection indices* $I(X, Y)$ of E-varieties X, Y of complementary dimensions. The main obstacle to define the intersection index is the infinity of the 0-dimensional E-variety. We define the *weak density* of 0-dimensional E-variety, which is analogous to the number of points of a 0-dimensional algebraic variety, and then use the weak density to define the intersection numbers.

Introduction, II. Ring of conditions

Definition. **E-varieties X and Y are said to be equivalent if**

$$\forall Z: \; I(X,Z) = I(Y,Z)$$

The set of equivalence classes of codimension k form a homogeneous component of degree k in the graded commutative semiring with operations '$+$' and '$\times$' defined as follows.

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Theorem. **Let $X \in \mathcal{X}$, $Y \in \mathcal{X}$ where $\mathcal{X}$, $\mathcal{Y}$ are two fixed equivalence classes. Then, for almost all $z \in \mathbb{C}^n$, *(1)* the equivalence class of $(z+X) \cup Y$ and *(2)* the equivalence class of $(z+X) \cap Y$ do not depend on the choice of $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. We denote these classes be $\mathcal{X} + \mathcal{Y}$ and by $\mathcal{X} \times \mathcal{Y}$**

Here we skip the definition of "almost all".

Even after all the definitions I have omitted, assertion (2) remains completely non-obvious. Its proof is mainly based on the application of tropical geometry.

Algebraic dimension, weak density, intersection index and BKK theorem (brief description).

We define the *algebraic codimension* $\mathrm{codim_a} X$ of E-variety X and the *weak density* $d_w(X)$ of 0-dimensional E-variety X. The weak density is the analog of the number of points of a 0-dimensional algebraic variety.

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Let $\mathrm{codim}_{\mathrm{a}} X_1 + \ldots + \mathrm{codim}_{\mathrm{a}} X_k = n$. Then there exists a *domain of relatively full measure* (I am not clarifying this concept here) D in $(\mathbb{C}^n)^k$ such that, for $(z_1, \ldots, z_k) \notin D$, the weak densities of E-varieties $(z_1 + X_1) \cap \ldots \cap (z_k + X_k)$ are the same.

This weak density is called the *intersection index* $I(X_1, \ldots, X_k)$.

The intersection index of E-varieties is not integer.

It turns out that, as in polynomial case, the *intersection index of n exponential hypersurfaces is equal to the mixed volume of their Newton polytopes.*

Algebraic dimension (definition), I

Let $G \subset \mathrm{Re}\,\mathbb{C}^n$ be a subgroup with finitely many generators. Assume that G contains a basis of $\mathbb{C}^n$. Let E_G denote the ring of exponential sums with support in G. Fix a basis $\alpha_1, \dots, \alpha_N$ of G. Then each exponential sum $f \in E_G$ can be expressed uniquely as a Laurent polynomial $f(z) = P(\mathrm{e}^{\alpha_1 z}, \dots, \mathrm{e}^{\alpha_N z})$ of N variables $\mathrm{e}^{\alpha_1 z}, \dots, \mathrm{e}^{\alpha_N z}$. If the E-variety X is given by equations $f_1 = \dots = f_k = 0$, where $f_i \in E_G$, then the corresponding algebraic variety in $(\mathbb{C} \setminus 0)^N$ denote by $M(X)$ and call it *the model of E-variety* X.

Definition. We define the algebraic codimension $\mathrm{codim_a} X$ **of E-variety** X **as** $\mathrm{codim}\, M(X)$**.**

The algebraic codimension $\mathrm{codim_a} X$ does not depend on the different choices such as group G and its basis $\alpha_1, \dots, \alpha_N$.

Algebraic dimension (definition), II

Assume that the E-variety X is given by equations $f = g = 0$. If f and g have no common divisor, then $\mathrm{codim}_{\mathrm{a}} X = 2$; otherwise $\mathrm{codim}_{\mathrm{a}} X = 1$. In particular the algebraic codimension of the point $0 \in \mathbb{C}$ treated as the E-variety given by the equations $\mathrm{e}^z - 1 = \mathrm{e}^{\sqrt{2}} - 1 = 0$ has algebraic codimension 2.
Thus the codimension of E-variety X as an analytic set can be smaller than $\mathrm{codim}_{\mathrm{a}} X$. Let (X, z) be the irreducible germ of an E-variety X at the point $z \in X$. If (X, z) has a smaller codimension than $\mathrm{codim}_{\mathrm{a}} X$, then the germ is said to be *atypical*. It is known that *each atypical germ of an E-variety lies in a proper affine subspace* of $\mathbb{C}^n$.
In particular *any atypical component of* 0*-dimensional E-variety in* $\mathbb{C}^2$ *is a complex affine line.*

Weak density (definition), I

The simplest example of 0-dimensional E-variety is the n-dimensional lattice $L \subset \operatorname{Im} \mathbb{C}^n$ (as the zero set $2\pi i \mathbb{Z}$ of ES $\mathrm{e}^z - 1$) or the shift of such lattice in $\mathbb{C}^n$ (as for $\mathrm{e}^z - w$). In this case the weak density is defined as the periodicity of L, that is $1/\mathrm{vol}_n(C)$ where C is the fundamental cube of L. The weak density of the union of translated lattices is the sum of their weak densities.

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$X \subset \mathbb{C}^n$ is called an ε-perturbation of the translated lattice $z + L$ if **(a)** X lies in the ε-neighbourhood $(z + L)_\varepsilon$ of this translated lattice, and **(b)** the ε-neighbourhood of each point $x \in z + L$ contains precisely one point from X.

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Definition. The weak density of a finite union of ε-perturbed lattices is the sum of their weak densities.

Let, as above, E_G be the ring of ESs with support in group G with the basis $\alpha_1, \dots, \alpha_N$. Each ES $f \in E_G$ is the Laurent polynomial $P(\mathrm{e}^{\alpha_1 z}, \dots, \mathrm{e}^{\alpha_N z})$. We define the action of $(\mathbb{C} \setminus 0)^N$ in E_G as follows: if $t = (c_1, \dots, c_N)$ then

$$t\colon P(\mathrm{e}^{\alpha_1 z}, \dots, \mathrm{e}^{\alpha_n z}) \mapsto P(c_1\,\mathrm{e}^{\alpha_1 z}, \dots, c_N\,\mathrm{e}^{\alpha_N z}).$$

Weak density (definition), II

We extend this action to E-varieties: if E-variety X given by equations $f_1 = \cdots = f_k = 0$ then E-variety tX is given by equations $tf_1 = \cdots = tf_k = 0$. Now we give the description of the assertion which gives the definition of weak density of any E-variety.

Theorem. **The torus $(\mathbb{C} \setminus 0)^N$ contains a set of "relatively full measure $\mathcal{U}$, such that the following is true. For each $t \in U$ exists a finite set of lattices in the space $\operatorname{Im} \mathbb{C}^n$, such that the toric shift tX can be approximated by unions of shifts of some finite set of lattices. The set of lattices depends on the connected component of $\mathcal{U}$ containing the point t. However, the weak densities of the union of lattices for all t are the same.**

THANKS FOR ATTENTION !