Computing the dimensions of the components of tropical prevarieties

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Abstract. The main goal of this work is the study of tropical recurrent sequences determined by various relations. For a set of tropical recurrent sequences described by tropical relations, D. Grigoriev put forward a hypothesis of stabilization of the maximum dimensions of the components of tropical prevarieties. This hypothesis has not been proven yet. As part of this work, for various recurrent sequences, the appropriate tropical prevarieties were examined using the gfan package in order to check Grigoriev's hypotises. The validity of such a hypothesis would make it possible to calculate the corresponding dimensions for a recurrent sequence for an arbitrary length.

Introduction

As part of this work, for various tropical recurrent relations, the corresponding tropical prevarieties were studied using the Gfan package in order to check the Grigoriev hypothesis about the stabilization of the maximum dimensions of the components, i.e the existence of a tropical analogue of the Hilbert polynomial. This hypothesis has not been proven yet. As part of this work, for various recurrent sequences, the appropriate tropical prevarieties were examined using the gfan package in order to check Grigoriev's hypotises.

In this work, the dimensions of the space of sequences are calculated in the cases of various recurrent relations. According to the calculated dimensions, the increase rate of the space of sequences relative to the number of elements in finite tropical sequences was revealed. Based on this regularity, hypotheses were made about the value of tropical entropy for various tropical recurrence relations. The calculations were made in the gfan package developed in 2005 by A. Jensen.

Gfan is a software package for calculating Gröbner fans and tropical varieties, developed in 2005 by A. Jensen, based on the algorithms in his dissertation [2]. The gfan package allows computing Gröbner bases, Gröbner fans, tropical prevarieties, varieties by given polynomials, and other objects of tropical geometry and the theory of Gröbner bases. It is currently the most powerful software tool for such calculations. Gfan is distributed as a standard Linux package and is part of the Debian distribution.

1. Basic objects of tropical math

Basic object of study is the *tropical semiring* $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$. If T is an ordered semi-group then T is a tropical semi-ring with inherited operations $\oplus := max$, $\otimes :=$ $+$. As a set this is just the real numbers $\mathbb{R}$, together with an extra element $-\infty$. In this semiring, the basic arithmetic operations of addition and multiplication of real numbers are redefined as follows:

$$x \oplus y := \max(x, y) \quad \text{and} \quad x \otimes y := x + y.$$

Many of the familiar axioms of arithmetic remain valid in tropical mathematics. For instance, both addition and multiplication are commutative. These two arithmetic operations are also associative, and the times operator takes $\otimes$ precedence when plus $\oplus$ and times $\otimes$ occur in the same expression. The distributive law holds for tropical addition and multiplication. [4]

Both arithmetic operations have an identity element. Minus infinity is the identity element for addition and zero is the identity element for multiplication. An important difference between the tropical semiring and classical math is that tropical addition is idempotent $x \oplus x = x$.

Let $x_1, \ldots, x_n$ be variables which represent elements in the tropical semiring $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$. By commutativity, we can sort the product and write tropical monomial in the usual notation, with the variables raised to exponents:

$$q(x_1, \ldots, x_n) = a \otimes x_1^{i_1} \otimes \cdots \otimes x_n^{i_n}.$$

A monomial represents a function from $\mathbb{R}^n$ to $\mathbb{R}$. When evaluating this function in classical arithmetic, what we get is a linear function:

$$q(x_1, \ldots, x_n) = a + i_1 \cdot x_1 + \cdots + i_n \cdot x_n.$$

A *tropical polynomial* is a finite linear combination of tropical monomials:

$$p(x_1, \ldots, x_n) = \bigotimes_j \left(a_j \otimes x_1^{i_{j_1}} \otimes \cdots \otimes x_n^{i_{j_n}} \right).$$

Here the coefficients a_j are real numbers and the exponents $i_{j_1}, \ldots, i_{j_n}$ are integers. Every tropical polynomial represents a function $\mathbb{R}^n \to \mathbb{R}$. When evaluating this function in classical arithmetic, what we get is the maximum of a finite collection of linear functions, namely

$$p(x_1, \ldots, x_n) = \max_j \left(a_j + i_{j_1} \cdot x_1 + \cdots + i_{j_n} \cdot x_n \right).$$

Definition 1. $x = (x_1, \ldots, x_n)$ is a *tropical zero* of p if maximum $\max_j q_j$ is attained for at least two different values of j.

Let some vector $w \in \mathbb{R}^n$ be given. We will use it as weight vector of some monomial ordering. And in this case, we allow negative values of the weigths. The initial form $\mathrm{in}_w(f)$ of a polynomial f is the highest monomials of this polynomial when the degrees of monomials are weighted by the vector w.For example, if $g = x + 2y + z + 1$, then $\mathrm{in}_{(0,0,1)}(g) = z$ and $\mathrm{in}_{(0,0,-1)}(g) = x + 2y + 1$.The highest monomials at some weight vectors may have more than one. This is part of the description of a tropical hypersurface.

Definition 2. *Tropical hypersurface* of the polynomial f is the set

$$\mathcal{T}(f) = \{w \in \mathbb{R}^b : \mathrm{in}_w(f) \text{ is not monomial}\}.$$

The tropical hypersurface is described in the same order space as the weight space. It is easy to see that if the weight vectors differ by a constant factor, then the weight orders are the same. That is, the membership of one point in the space of a tropical hypersurface entails the membership of the ray on which this point lies.

The connection between the concept of a tropical hypersurface and tropical mathematics lies in the process of tropicalization. Tropicalization is the transition from objects of classical mathematics to objects of tropical mathematics, which is carried out as follows: classical addition, multiplication and exponentiation are replaced by their tropical counterparts, the coefficients at monomials are assumed to be equal to zero.

Definition 3. *Tropical prevariety* of a system of polynomials $f_1, \ldots, f_n$ is the finite intersection of tropical hypersurfaces

$$\mathcal{T}(f_1) \cap \cdots \cap \mathcal{T}(f_n).$$

2. Tropical recurrent sequences

A classical linear recurrent sequence $\{z_j\}_{j\in\mathbb{Z}}$ satisfies conditions $\sum_{0\le i\le n} a_i z_{i+k} = 0, k \in \mathbb{Z}, a_0 \neq 0, a_n \neq 0$. A remarkable property of classical linear recurrent sequences is as follows: since the last coefficient a_n is not equal to zero, then if you calculate all z up to z_i, you can uniquely calculate z_{i+1} by substituting the corresponding k into the formula. This property is satisfied, since in classical arithmetic there are elements inverse in addition.

Definition 4. $y = y_i \in (\mathbb{R} \cup \{-\infty\})_{j\in\mathbb{Z}}$ is a *tropical recurrent sequence* if it satisfies conditions

$$\max_{0\le i\le n} (a_i + y_{k+i}), \quad k \in \mathbb{Z}, \quad a_0 > -\infty, a_n > -\infty. \tag{1}$$

The fulfillment of this condition means reaching the maximum in two or more tropical terms $a_i + y_{k+i}$.

The main difference between tropical recurrent sequences and classical ones is that each subsequent term, knowing the previous ones, is not always uniquely determined. Tropical recurrent sequences can be either periodic or non-periodic.

A sequence y is called periodic if $\exists d > 0$ such that $y_i - d_{i \in \mathbb{Z}}$ satisfies the tropical recurrence conditions 1.

Periodic recurrent sequences are in a sense trivial, since they correspond to classical recurrent sequences. The presence of non-periodic sequences is a tropical effect, and it is this presence that is the reason for the increase in the number of recurrent sequences with their length. To define tropical entropy, we introduce the concept of finite tropical recurrent sequences.

$y = (y_0, \dots, y_s) \in (\mathbb{R} \cup \{-\infty\})^{s+1}$ is a *finite tropical recurrent sequence* if it satisfies conditions

$$\max_{0 \leq i \leq n} (a_i + y_{k+i}), \quad k \in \{0, 1, \dots, s-n\}, \quad a_0 > -\infty, a_n > -\infty.$$

Definition 5. Denote by $D_s := D_s(a) \in (\mathbb{R} \cup \{-\infty\})^{s+1}$ the set of sequences satisfying vector a, and denote by $d_s := \dim D_s$. *Tropical entropy* is the limit $H(a) := \lim\limits_{s \to \infty} \frac{d_s}{s}$.

In the paper [1] D. Grigoriev proved the existence of entropy, as well as some properties.

3. Computing of tropical prevarieties corresponding to tropical recurrent sequences

Since the tropical entropy is a limit, in this paper reasonable hypotheses are given, what it can be equal to. To calculate the hypothetical tropical entropy, the vector a is associated with a system of $n - s + 1$ linear tropical equations with $s + 1$ unknowns, then the tropical prevariety of the system of equations are calculated. The gfan package is used to compute tropical prevarieties. The GFAN package computes tropical prevarieties only for polynomials with zero coefficients. For non-zero coefficients, a parametrization is introduced, which is discussed in detail in the GFAN manual [3] when calculating tropical curves.

From the computed tropical prevariety, one can find d_s. With a series of calculations with different s, you can find a pattern of growth in dimension and draw a conclusion about the hypothetical tropical entropy.

Using linear transformations, the vector $a = (a_0, \dots, a_n)$ can be associated with the vector $b = (0, b_1, \dots, b_{n-1}, 0)$. It is technically easier to consider cases in which $a_0 = 0$ and $a_n = 0$. The calculations were done for all such vectors of length n=3, presented in Table 1.

Conclusion

Computations of tropical prevarieties are performed to study the asymptotics of d_s and the conduct of the tropical entropy for various cases of a vector a of length $n = 3$. All hypothetical values of tropical entropy satisfy the properties proved in [1]. As a continuation of this work, it is proposed to do the following:

1. Computing of tropical prevarieties corresponding to tropical recurrent sequences of vectors of greater length.
2. Computing of tropical prevarieties for systems of non-recurrent equations.
3. Development of an interface for tropical computing.

a\\s	d_5	d_6	d_7	d_8	d_9	d_{10}	d_{11}	d_{12}	d_{13}	d_{14}	d_{15}	d_{16}	d_{17}	d_{18}	H(a)
(0,0,0,0)	4	4	4	5	6	6	6	7	8	8	8	9	10	10	1/2
(0,1,-1,0)	4	5	5	6	6	7	7	8	8	9	9	10	10	11	1/2
(0,-1,-1,0)	3	4	5	5	5	5	5	6	7	7	7	7	7	8	1/3
(0,1,2,0)	4	4	5	5	5	6	6	6	7	7	7	8	8	8	1/3
(0,-1,-2,0)	4	5	5	5	6	6	6	7	7	7	8	8	8	9	1/3
(0,-1,-3,0)	3	4	4	4	5	5	5	6	6	6	7	7	7	8	1/3
(0,-2,-3,0)	3	4	5	5	5	5	5	6	7	7	7	7	7	8	1/3
(0,-1,-∞, 0)	3	4	4	4	5	5	5	6	6	6	7	7	7	8	1/3
(0,0,-∞, 0)	3	3	4	4	4	4	5	5	5	6	6	6	6	7	2/7
(0,1,3,0)	4	4	4	4	5	5	5	5	6	6	6	6	7	7	1/4
(0,-1,2,0)	4	4	4	4	5	5	5	5	6	6	6	6	7	7	1/4
(0,1,-2,0)	3	4	4	4	4	5	5	5	5	6	6	6	6	7	1/4
(0,-1,0,0)	4	4	5	5	5	5	6	6	6	6	7	7	7	7	1/4
(0,1,0,0)	4	4	4	4	5	5	5	5	6	6	6	6	7	7	1/4
(0,1,-∞, 0)	3	4	4	4	4	5	5	5	5	6	6	6	6	7	1/4
(0,1,1,0)	3	3	3	3	3	3	3	3	3	3	3	3	3	3	0
(0,2,3,0)	3	3	3	3	3	3	3	3	3	3	3	3	3	3	0
(0,-∞, $-\infty$, 0)	3	3	3	3	3	3	3	3	3	3	3	3	3	3	0

TABLE 1. Hypothetical tropical entropy for $n = 3$

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