Computing the dimensions of the components of tropical prevarieties

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1. **Basic objects of tropical math**
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    - Tropical polynomial
    - Tropical prevariety
2. **Tropical recurrent sequences**
    - Classical recurrent sequences
    - Tropical recurrent sequences
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Introduction

The main goal of this work is the study of tropical recurrent sequences. For a set of tropical recurrent sequences determined by tropical relations, D. Grigoriev put forward a hypothesis of stabilization of the maximum dimensions of the components of tropical prevarieties, i.e the existence of a tropical analogue of the Hilbert polynomial. This hypothesis has not been proven yet. As part of this work, for various recurrent sequences, the appropriate tropical prevarieties were examined using the gfan package in order to check Grigoriev's hypotises.

Gfan is a software package developed in 2005 by A. Jensen, based on the algorithms in his dissertation. The gfan package allows computing Gröbner bases, Gröbner fans, tropical prevarieties, varieties by given polynomials, and other objects of tropical geometry and the theory of Gröbner bases. It is currently the most powerful software tool for such computations.

Tropical semiring

Definition

The *tropical semiring* is a semiring $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ with tropical operations:

$$x \oplus y := \max(x, y) \quad \text{and} \quad x \otimes y := x + y.$$

Examples:

$$5 \oplus 2 = 5, \quad 5 \otimes 2 = 7.$$

A main difference between the tropical semiring and classical math is that tropical addition is idempotent $x \oplus x = x$. It is also important that there is no subtraction operation in the tropical semiring. The equation $a \oplus x = -\infty$ has no solution for any a except minus infinity itself.

Tropical polynomial

Tropical monomial

Let $x_1, \ldots, x_n$ be variables which represent elements in the tropical semiring $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$. By commutativity, we can sort the product and write *tropical monomial* in the usual notation:

$$q(x_1, \ldots, x_n) = a \otimes x_1^{i_1} \otimes \cdots \otimes x_n^{i_n} = a + i_1 \cdot x_1 + \cdots + i_n \cdot x_n.$$

Tropical polynomial

A *tropical polynomial* is a finite linear combination of tropical monomials:

$$p(x_1, \ldots, x_n) = \bigoplus_j \left(a_j \otimes x_1^{i_{j_1}} \otimes \cdots \otimes x_n^{i_{j_n}}\right) = \max_j \left(a_j + i_{j_1} x_1 + \cdots + i_{j_n} x_n\right)$$

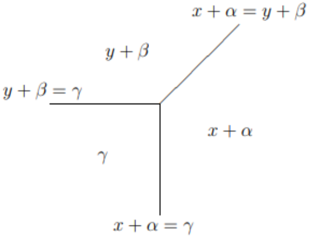
Tropical line

Tropical root

$x = (x_1, \ldots, x_n)$ is a *tropical root* of p if maximum of tropical monomials q_j is attained for at least two different values of j.

For example, consider the tropical line:

$$f(x, y) = \alpha \otimes x \oplus \beta \otimes y \oplus \gamma = \max(\alpha + x, \beta + y, \gamma)$$

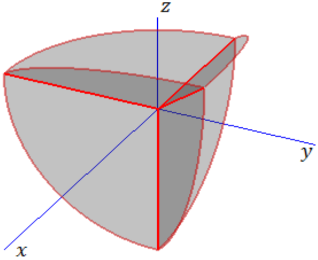


Tropical hypersurface

Definition

Tropical hypersurface $\mathcal{T}(f)$ of the polynomial f is the set of tropical roots of tropical polynomial $Trop(f)$. *Tropicalization* $Trop(f)$ is the transition from objects of classical mathematics to objects of tropical, and the coefficients at monomials are assumed to be equal to zero.

For example, consider the tropical hypersurface of polynomial
$g = x + 2y + z + 1$. $Trop(g) = 0 \otimes x \oplus 0 \otimes y \oplus 0 \otimes z \oplus 0 = \max(x, y, z, 0)$.

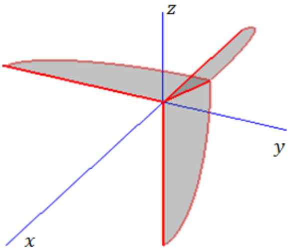


Tropical prevariety

Definition

Tropical prevariety of a system of polynomials $f_1, \ldots, f_n$ is the finite intersection of tropical hypersurfaces $\mathcal{T}(f_1) \cap \cdots \cap \mathcal{T}(f_n)$.

For example, consider the tropical hypersurface of polynomial $g = x + 2y + z + 1$ and $h = x + y + 2z$.
$Trop(g) = 0 \otimes x \oplus 0 \otimes y \oplus 0 \otimes z \oplus 0 = \max(x, y, z, 0)$;
$Trop(h) = 0 \otimes x \oplus 0 \otimes y \oplus 0 \otimes z = \max(x, y, z)$.



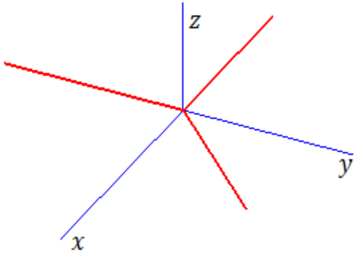
Tropical variety

Definition

Let I an ideal generated from polynomials $\langle f_1, \ldots, f_n \rangle$. *Tropical variety* $\mathcal{T}(I)$ of an ideal I is the intersection of tropical hypersurfaces $\mathcal{T}(f)$ for all polynomials $f \in I$

$$\mathcal{T}(I) = \bigcap_{f \in I} \mathcal{T}(f)$$

For example, consider the tropical variety of ideal $I = \langle g, f \rangle = \langle x + 2y + z + 1, x + y + 2z \rangle$.



Classical linear recurrent sequences

A classical linear recurrent sequence $\{z_j\}_{j\in\mathbb{Z}}$ satisfies conditions $\sum_{0\leq i\leq n} a_i z_{i+k} = 0, k \in \mathbb{Z}, a_0 \neq 0, a_n \neq 0$. A remarkable property of classical linear recurrent sequences is as follows: since the last coefficient a_n is not equal to zero, then if you calculate all z up to z_i, you can uniquely calculate z_{i+1} by substituting the corresponding k into the formula. This property is satisfied, since in classical arithmetic there are elements inverse in addition.

Tropical recurrent sequences

Definition

$y = y_i \in (\mathbb{R} \cup \{-\infty\})_{j \in \mathbb{Z}}$ is a *tropical recurrent sequence* if it satisfies conditions

$$\max_{0 \le i \le n} (a_i + y_{k+i}), \quad k \in \mathbb{Z}, \quad a_0 > -\infty, a_n > -\infty.$$

The fulfillment of this condition means reaching the maximum in two or more tropical terms $a_i + y_{k+i}$.

The main difference between tropical recurrent sequences and classical sequences is that each subsequent term, knowing the previous ones, is not always uniquely determined. Tropical recurrent sequences can be either periodic or non-periodic.

Tropical entropy

Finite tropical recurrent sequence

$y = (y_0, \ldots, y_s) \in (\mathbb{R} \cup \{-\infty\})^{s+1}$ is a *finite tropical recurrent sequence* if it satisfies conditions

$$\max_{0 \le i \le n} (a_i + y_{k+i}), \quad k \in \{0, 1, \ldots, s-n\}, \quad a_0 > -\infty, a_n > -\infty.$$

Definition

Denote by $D_s := D_s(a) \in (\mathbb{R} \cup \{-\infty\})^{s+1}$ the set of sequences satisfying vector a, and denote by $d_s := \dim D_s$. *Tropical entropy* is the limit $H(a) := \lim\limits_{s \to \infty} \frac{d_s}{s}$.

Computations in Gfan

Since the tropical entropy is a limit, in this work reasonable hypotheses are given, what it can be equal to. To calculate the hypothetical tropical entropy, the vector a is associated with a system of $n - s + 1$ linear tropical equations with $s + 1$ unknowns, then the tropical prevariety of the system of equations are calculated. The gfan package is used to compute tropical prevarieties.

From the computed tropical prevariety, one can find d_s. With a series of calculations with different s, can find a pattern of growth in dimension and draw a conclusion about the hypothetical tropical entropy.

Using linear transformations, the vector $a = (a_0, \dots, a_n)$ can be associated with the vector $b = (0, b_1, \dots, b_{n-1}, 0)$. It is technically easier to consider cases in which $a_0 = 0$ and $a_n = 0$.

Computing prevariety for $a = (0, 0, 0)$, $s = 15$

```
in15 ×   out15 ×
Q[y00,y01,y02,y03,y04,y05,y06,y07,y08,y09,y10,y11,y12,y13,y14,y15]
{
y00+y01+y02, y01+y02+y03, y02+y03+y04, y03+y04+y05, y04+y05+y06, y05+y06+y07,
y06+y07+y08, y07+y08+y09, y08+y09+y10, y09+y10+y11, y10+y11+y12, y11+y12+y13,
y12+y13+y14, y13+y14+y15}
```

```
in15 ×   out15 ×
_application fan
_version 2.2
_type SymmetricFan

AMBIENT_DIM
16

DIM
7

LINEALITY_DIM
0

RAYS
-1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 # 0
0 -1 0 0 0 0 0 0 0 0 0 0 0 0 0 0        # 1
0 0 -1 0 0 0 0 0 0 0 0 0 0 0 0 0        # 2
0 0 0 -1 0 0 0 0 0 0 0 0 0 0 0 0        # 3
0 0 0 0 -1 0 0 0 0 0 0 0 0 0 0 0        # 4
0 0 0 0 0 -1 0 0 0 0 0 0 0 0 0 0        # 5
0 0 0 0 0 0 -1 0 0 0 0 0 0 0 0 0        # 6
0 0 0 0 0 0 0 -1 0 0 0 0 0 0 0 0        # 7
0 0 0 0 0 0 0 0 -1 0 0 0 0 0 0 0        # 8
0 0 0 0 0 0 0 0 0 -1 0 0 0 0 0 0        # 9
0 0 0 0 0 0 0 0 0 0 -1 0 0 0 0 0        # 10
0 0 0 0 0 0 0 0 0 0 0 -1 0 0 0 0        # 11
0 0 0 0 0 0 0 0 0 0 0 0 -1 0 0 0        # 12
0 0 0 0 0 0 0 0 0 0 0 0 0 -1 0 0        # 13
0 0 0 0 0 0 0 0 0 0 0 0 0 0 -1 0        # 14
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 -1        # 15
0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 # 16
```

A sequence with the maximal dimension satisfies vector $a = (0, 0, 0)$:

$$y = (r - c_0, r, r, r - c_3, r, r, r - c_6, r, r, r - c_9, r, r, r - c_{12}, r, r, r - c_{15}),\ d_{15} = 7.$$

Computing prevariety for $a = (0, -1, 0)$, $s = 15$

in15 ×

```
Q[t,y00,y01,y02,y03,y04,y05,y06,y07,y08,y09,y10,y11,y12,y13,y14,y15]
{
y00+t^1y01+y02, y01+t^1y02+y03, y02+t^1y03+y04, y03+t^1y04+y05, y04+t^1y05+y06,
y05+t^1y06+y07, y06+t^1y07+y08, y07+t^1y08+y09, y08+t^1y09+y10, y09+t^1y10+y11,
y10+t^1y11+y12, y11+t^1y12+y13, y12+t^1y13+y14, y13+t^1y14+y15}
```

in15 × out15 ×

```
_application fan
_version 2.2
_type SymmetricFan

AMBIENT_DIM
17

DIM
7

LINEALITY_DIM
1

RAYS
-2 -1 1 -1 1 -1 1 -1 1 -1 1 -1 1 -1 1 -1 1        # 0
-2 1 -1 1 -1 1 -1 1 -1 1 -1 1 -1 1 -1 1 -1        # 1
0 -15 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1      # 2
0 1 -15 1 1 1 1 1 1 1 1 1 1 1 1 1 1      # 3
0 1 1 -15 1 1 1 1 1 1 1 1 1 1 1 1 1      # 4
0 1 1 1 -15 1 1 1 1 1 1 1 1 1 1 1 1      # 5
0 1 1 1 1 -15 1 1 1 1 1 1 1 1 1 1 1      # 6
0 1 1 1 1 1 -15 1 1 1 1 1 1 1 1 1 1      # 7
0 1 1 1 1 1 1 -15 1 1 1 1 1 1 1 1 1      # 8
0 1 1 1 1 1 1 1 -15 1 1 1 1 1 1 1 1      # 9
0 1 1 1 1 1 1 1 1 -15 1 1 1 1 1 1 1      # 10
0 1 1 1 1 1 1 1 1 1 -15 1 1 1 1 1 1      # 11
0 1 1 1 1 1 1 1 1 1 1 -15 1 1 1 1 1      # 12
0 1 1 1 1 1 1 1 1 1 1 1 -15 1 1 1 1      # 13
0 1 1 1 1 1 1 1 1 1 1 1 1 -15 1 1 1      # 14
0 1 1 1 1 1 1 1 1 1 1 1 1 1 -15 1 1      # 15
0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 -15 1      # 16
0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 -15      # 17
```

A sequence with the maximal dimension satisfies vector $a = (0, -1, 0)$:

$$y = (r - c_0, r+1, r, r+1, r - c_4, r+1, r, r+1, r - c_8,$$

$$r+1, r, r+1, r - c_{12}, r+1, r, r+1), d_{15} = 5$$

in10 ×

```
Q[t1,t2,y00,y01,y02,y03,y04,y05,y06,y07,y08,y09,y10]
{
y00+t1y01+t2y02+y03, y01+t1y02+t2y03+y04, y02+t1y03+t2y04+y05,
y03+t1y04+t2y05+y06, y04+t1y05+t2y06+y07, y05+t1y06+t2y07+y08,
y06+t1y07+t2y08+y09, y07+t1y08+t2y09+y10}
```

```
RAYS
-22 -11 1 -10 12 1 -10 12 1 -10 12 1 -10        # 0
-22 -11 10 -1 -12 10 -1 -12 10 -1 -12 10 -1      # 1
-11 -22 -10 1 12 -10 1 12 -10 1 12 -10 1         # 2
-11 -22 -1 10 -12 -1 10 -12 -1 10 -12 -1 10      # 3
-11 0 -8 3 -8 -8 3 3 3 3 3 3 3  # 4
-11 0 -5 6 -5 -5 6 6 6 -5 6 -5 -5       # 5
-11 0 3 -8 3 -8 -8 3 3 3 3 3 3  # 6
-11 0 3 3 -8 3 -8 -8 3 3 3 3 3  # 7
-11 0 3 3 3 -8 3 -8 -8 3 3 3 3  # 8
-11 0 3 3 3 3 3 -8 3 -8 -8 3 3  # 9
-11 0 3 3 3 3 3 3 -8 3 -8 -8 3  # 10
-11 0 3 3 3 3 3 3 3 -8 3 -8 -8  # 11
-11 11 -28 -17 -39 -28 -17 -6 5 16 27 38 49      # 12
-11 11 -7 4 -18 -7 -29 -18 -7 4 15 26 37         # 13
-11 11 -1 -23 -12 -34 -23 -12 -1 10 21 32 43     # 14
-11 11 8 19 -3 8 -14 -3 -25 -14 -3 8 19 # 15
-11 11 17 -5 6 -16 -5 -27 -16 -5 6 17 28         # 16
-11 11 17 28 6 17 -5 6 -16 -5 -27 -16 -5         # 17
-11 11 29 7 18 -4 7 -15 -4 -26 -15 -4 7 # 18
-11 11 35 13 24 2 13 -9 2 -20 -9 -31 -20         # 19
-2 -1 -1 1 0 -1 1 0 -1 1 0 -1 1 # 20
-1 -2 1 -1 0 1 -1 0 1 -1 0 1 -1 # 21
-1 0 0 0 0 0 0 0 0 0 0 0 0      # 22
0 -11 -8 -8 3 -8 3 3 3 3 3 3 3  # 23
0 -11 -5 -5 6 -5 6 6 6 -5 -5 6 -5       # 24
0 -11 3 -8 -8 3 -8 3 3 3 3 3 3  # 25
0 -11 3 3 -8 -8 3 -8 3 3 3 3 3  # 26
0 -11 3 3 3 3 -8 -8 3 -8 3 3 3  # 27
0 -11 3 3 3 3 3 -8 -8 3 -8 3 3  # 28
0 -11 3 3 3 3 3 3 -8 -8 3 -8 3  # 29
0 -11 3 3 3 3 3 3 3 -8 -8 3 -8  # 30
0 -1 0 0 0 0 0 0 0 0 0 0 0      # 31
0 0 -10 1 1 1 1 1 1 1 1 1 1     # 32
0 0 1 -10 1 1 1 1 1 1 1 1 1     # 33
0 0 1 1 -10 1 1 1 1 1 1 1 1     # 34
0 0 1 1 1 -10 1 1 1 1 1 1 1     # 35
0 0 1 1 1 1 -10 1 1 1 1 1 1     # 36
0 0 1 1 1 1 1 -10 1 1 1 1 1     # 37
0 0 1 1 1 1 1 1 -10 1 1 1 1     # 38
0 0 1 1 1 1 1 1 1 -10 1 1 1     # 39
0 0 1 1 1 1 1 1 1 1 -10 1 1     # 40
0 0 1 1 1 1 1 1 1 1 1 -10 1     # 41
0 0 1 1 1 1 1 1 1 1 1 1 -10     # 42
11 -11 -20 -31 -9 -20 2 -9 13 2 24 13 35        # 43
11 -11 -5 -16 -27 -5 -16 6 -5 17 6 28 17        # 44
11 -11 7 -4 -15 -26 -4 -15 7 -4 18 7 29 # 45
11 -11 19 8 -3 -14 -25 -3 -14 8 -3 19 8 # 46
11 -11 28 17 6 -5 -16 -27 -5 -16 6 -5 17        # 47
11 -11 37 26 15 4 -7 -18 -29 -7 -18 4 -7        # 48
11 -11 43 32 21 10 -1 -12 -23 -34 -12 -23 -1    # 49
11 -11 49 38 27 16 5 -6 -17 -28 -39 -17 -28     # 50
11 22 -29 -40 -51 -62 -40 -18 4 26 48 70 92     # 51
11 22 -8 -19 -30 -41 -52 -30 -8 14 36 58 80     # 52
11 22 10 -1 -12 -23 -34 -45 -23 -1 21 43 65     # 53
11 22 25 14 3 -8 -19 -30 -41 -19 3 25 47        # 54
11 22 37 26 15 4 -7 -18 -29 -40 -18 4 26        # 55
11 22 46 35 24 13 2 -9 -20 -31 -42 -20 2        # 56
11 22 52 41 30 19 8 -3 -14 -25 -36 -47 -25      # 57
22 11 -25 -47 -36 -25 -14 -3 8 19 30 41 52      # 58
22 11 2 -20 -42 -31 -20 -9 2 13 24 35 46        # 59
22 11 26 4 -18 -40 -29 -18 -7 4 15 26 37        # 60
22 11 47 25 3 -19 -41 -30 -19 -8 3 14 25        # 61
22 11 65 43 21 -1 -23 -45 -34 -23 -12 -1 10     # 62
22 11 80 58 36 14 -8 -30 -52 -41 -30 -19 -8     # 63
22 11 92 70 48 26 4 -18 -40 -62 -51 -40 -29     # 64
```

In this case, 5573 cones of dimensions from 2 to 7 were obtained, which is difficult to recognize which vector belongs to which cone. For example, the vector $a = (0, -1, -2, 0)$ corresponds to the cone formed by the rays 2, 32, 33, 38, 41, 42 or 3, 32, 34, 37, 40, 41 et al.

All results for vectors $a = (0, a_1, a_2, 0)$

a\s	5	6	7	8	9	10	11	12	13	14	15	16	17	18	H(a)
(0,0,0,0)	4	4	4	5	6	6	6	7	8	8	8	9	10	10	1/2
(0,1,-1,0)	4	5	5	6	6	7	7	8	8	9	9	10	10	11	1/2
(0,-1,-1,0)	3	4	5	5	5	5	5	6	7	7	7	7	7	8	1/3
(0,1,2,0)	4	4	5	5	5	6	6	6	7	7	7	8	8	8	1/3
(0,-1,-2,0)	4	5	5	5	6	6	6	7	7	7	8	8	8	9	1/3
(0,-1,-3,0)	3	4	4	4	5	5	5	6	6	6	7	7	7	8	1/3
(0,-2,-3,0)	3	4	5	5	5	5	5	6	7	7	7	7	7	8	1/3
(0,-1,-∞,0)	3	4	4	4	5	5	5	6	6	6	7	7	7	8	1/3
(0,0,-∞,0)	3	3	4	4	4	4	5	5	5	6	6	6	6	7	2/7
(0,1,3,0)	4	4	4	4	5	5	5	5	6	6	6	6	7	7	1/4
(0,-1,2,0)	4	4	4	4	5	5	5	5	6	6	6	6	7	7	1/4
(0,1,-2,0)	3	4	4	4	4	5	5	5	5	6	6	6	6	7	1/4
(0,-1,0,0)	4	4	5	5	5	5	6	6	6	6	7	7	7	7	1/4
(0,1,0,0)	4	4	4	4	5	5	5	5	6	6	6	6	7	7	1/4
(0,1,-∞,0)	3	4	4	4	4	5	5	5	5	6	6	6	6	7	1/4
(0,1,1,0)	3	3	3	3	3	3	3	3	3	3	3	3	3	3	0
(0,2,3,0)	3	3	3	3	3	3	3	3	3	3	3	3	3	3	0
(0,-∞,$-\infty$,0)	3	3	3	3	3	3	3	3	3	3	3	3	3	3	0

Conclusion

Computations of tropical prevarieties are performed to study the asymptotics of d_s and the conduct of the tropical entropy for various cases of a vector a of length $n = 3$. All hypothetical values of tropical entropy satisfy the properties proved by D.Grigoriev. As a continuation of this work, it is proposed to do the following:

1. Computing of tropical prevarieties corresponding to tropical recurrent sequences of vectors of greater length.
2. Computing of tropical prevarieties for systems of non-recurrent equations.
3. Development of an interface for tropical computing.