GInv: software for calculation of Gröbner involutive basis

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Abstract. The open source software GINV implements the Gröbner bases method for systems of equations. In the report, a new revised version of GInv will be presented. We use this system for analytical study of cubature formulas on a sphere known as Popov's problem.

The study of systems of nonlinear equations in modern computer algebra systems is based on the calculation of Gröbner bases of ideals generated by the left-hand sides of the equations of these systems. The implementation of the Buchberger algorithm, which came to this system from the Singular system, is used. Buchberger's algorithm is the oldest, the basic version of Buchberger's algorithm leaves a lot of freedom in carrying out the computational process, thus considerable improvements are obtained by implementing criteria for reducing the number of S-polynomials to be actually considered (e.g., by applying the product criterion or the chain criterion).

In the late 1990s, involutive algorithms [1, 2] were proposed as an alternative to the Buchberger algorithm and implemented in the GInv system. On the initiative of V.P. Gerdt and Yu.A. Blinkov in 2005, the GInv project (`http://invo.jinr.ru`) was founded, within which software was developed for calculating involutive bases, written as a C++ module for the Python language. Recently, this system has been significantly revised by one of the authors of this paper, the new version is in the public access and is available at `https://github.com/blinkovua/GInv`. In the new version of the GInv system, dynamic memory reallocation mechanisms have been added, which allow speeding up calculations significantly, up to several times. The proposed approach is fundamentally different from other algorithms known under the general name 'garbage collection' [3] and is based on the implementation of object-oriented programming in C++.

A good demonstration of the achievements of the GInv system was an analytical study of cubature formulas on a sphere known as Popov's problem. The development of the theory of cubature formulas on the sphere that are invariant under transformations of finite symmetry groups of regular polyhedra was the subject of recent studies by A.S. Popov [4]. In Popov's works the solution of the problem is reduced to the study of a system of nonlinear algebraic equations, which was then solved numerically using the computer facilities of the Siberian Supercomputer Center. We investigate the problem of finding the weights and nodes of cubature formulas of a given order on a unit sphere that are invariant under the rotation groups of the icosahedron (described by A.S. Popov's in [5]) using free-access computer algebra systems, namely, the popular general-purpose system Sage (`https://www.sagemath.org`) and the GInv system.

Popov's algorithm for reducing the problem to a system of nonlinear equations is implemented in Sage. For approximation orders 19 and 20, the set of solutions is described using standard tools of the Sage system. For order 23, Popov's problem could not be solved in Sage, since the system could not calculate the Gröbner basis of the ideal for the system of equations describing Popov's problem in this case in a few days. However, the GInv system successfully coped with this task; it turned out that the basis polynomials have extremely large integer coefficients. Further, using the well-known Gröbner basis, it was possible to completely describe the set of solutions to the Popov problem in Sage. The exact solutions found using computer algebra systems are compared with the solutions found numerically by Popov. In particular, a new solution of the Popov problem is found for the order of approximation equal to 23.

It should be emphasized that this test is not artificial, specially invented for testing computer algebra systems, and the results themselves are of general scientific interest. It is impossible not to note the significant difference between the parameters characterizing the calculation of the Gröbner basis when studying the Popov problem and standard tests (`http://invo.jinr.ru/ginv/benchmark.html`): the coefficients of the basis polynomials turn out to be huge, and the degrees on the contrary, remain small.

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