Symplectic Structures in Finite Quantum Mechanics and Generalized Clifford Algebras

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Abstract. In the Hamiltonian formulation of classical mechanics, the state of a system is described by pairs of conjugate variables q and p called *positions* and *momenta*. These pairs form an even-dimensional *symplectic* manifold.

When constructing quantum mechanics, the variables q and p are naturally replaced with Hermitian operators $\hat{q}$ and $\hat{p}$ that must satisfy the Heisenberg *canonical commutation relation* $[\hat{q}, \hat{p}] = \mathbf{i}\hbar$, without which it is impossible to describe quantum interference. However, this relation is not fundamental. Being, in fact, an infinitesimal approximation of a more fundamental relation, it can only be realized in an infinite-dimensional Hilbert space. Replacing the conjugate *Hermitian* operators $\hat{q}$ and $\hat{p}$ with a pair of *unitary* operators Q and P, Hermann Weyl constructed a canonical commutation relation

$$QP = \omega PQ,\ \omega = \mathrm{e}^{2\pi\mathbf{i}/N}.$$

Weyl proved that the matrices Q and P are generators of a projective representation of $\mathbb{Z}_N \times \mathbb{Z}_N$ in the N-dimensional Hilbert space and coincide with "the shift and clock matrices" discovered by J.J. Sylvester in the 19th century. The shift matrix Q is the matrix of cyclic permutation of N elements. The clock matrix P is simply the diagonal form of the matrix Q.

The orthonormal bases associated with the matrices Q and P are *mutually unbiased bases*, a concept that is a deep quantum version of the symplectic conjugation introduced by J. Schwinger.

The matrices Q and P generate a structure called the *generalized Clifford algebra*. This structure is quite non-trivial and even in the simplest case $N = 2$ allows one to describe quaternions, three-dimensional rotations, spin-$\frac{1}{2}$ particles, etc. Thus, having only single cyclic permutation of N elements, using purely mathematical means — linear algebra, projective representations and central group extensions — we get rich tools for studying quantum-mechanical problems.

The fundamental role of the cyclic permutation matrix Q agrees well with the *finite quantum mechanics* [1–3] describing unitary evolutions by permutations, which are always products of cyclic permutations.

References

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