Symplectic Structures in Finite Quantum Mechanics and Generalized Clifford Algebras

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Standard Quantum Mechanics

- Hilbert space $\mathcal{H}$ over field $\mathcal{F} = \mathbb{C}$
  $\dim \mathcal{H} = \infty$ (mostly in physics) or $\dim \mathcal{H} = N$ (in quantum informatics)

- Pure state is a ray $\alpha|\psi\rangle \in \mathcal{H},\ \alpha \in \mathcal{F}^*$

- Mixed state is a density matrix ρ
  - hermitian: $\rho = \rho^*$
  - positive semi-definite: $\rho \geq 0$
  - normed: $\operatorname{tr} \rho = 1$

  for pure state $\rho = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}$ is rank-1 projector

- Closed system (universe) is a pure state in a unitary evolution

- Subsystems in universe are given by factoring $\mathcal{H}$ into a tensor product

Problems: explaining within this framework the emergence of observed structures and phenomena: geometry, gravity, dynamics, quantum correlations, etc.

David Hilbert. *On the infinite*

"Our principal result is that the infinite is nowhere to be found in reality. It neither exists in nature nor provides a legitimate basis for rational thought — a remarkable harmony between being and thought."

Constructive version of QM: "Finite Quantum Mechanics"

- Set of primary ("ontic") objects: $\Omega \simeq \{0, \ldots, \mathcal{N}-1\}$
- Permutation group on Ω: $G \leq \mathrm{Sym}(\Omega)$
- Permutation (matrix) representation $\mathcal{M}$:

$$\mathcal{M}(g)_{k,\ell} = \delta_{kg,\ell}, \quad g \in \mathrm{Sym}(\Omega), \quad k, \ell \in \Omega$$

- Hilbert space $\mathcal{H}_{\mathcal{N}-1}$ is the standard subspace of representation $\mathcal{M}$
- Field $\mathcal{F} = \mathrm{Frac}(\mathbb{N}[\omega]) = \mathbb{Q}(\omega) < \mathbb{C}, \quad \omega = \mathrm{e}^{2\pi\mathrm{i}/\mathrm{Exp}(G)}$

Tom Banks: "[This version of QM] can accurately reproduce all of the results of conventional quantum mechanics."

Evolution of a closed quantum system

- Standard QM: continuous time $t \in \mathbb{R}$
  one-parameter unitary group $U_t = \mathrm{e}^{-\mathrm{i}Ht}$
  generated by Hamiltonian H
- Finite QM: discrete time $t \in \mathbb{Z}$
  one-parameter unitary group U^t
  generated by permutation matrix $U = \mathcal{M}(g)$
  permutation $g \in \mathrm{Sym}(\Omega)$ is a product of disjoint cycles:

$$g = (c_1)\cdots(c_k)\cdots(c_K)\,, \quad c_k^{N_k} = \mathbf{1}, \quad \sum_{k=1}^{K} N_k = \mathcal{N}$$

$$\Downarrow$$

$$U = \begin{pmatrix} \mathcal{M}(c_1) & 0 & \cdots & 0 \\ 0 & \mathcal{M}(c_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{M}(c_K) \end{pmatrix}$$

Generator of cyclic group $\mathbb{Z}_N$: cyclic permutation $c = (0, 1, \ldots, N-1)$

- Ontic basis: $\mathcal{B}_o = \{|0\rangle, |1\rangle, \ldots, |N-1\rangle\}$
- Circular shift matrix: $\hat{\mathrm{A}}\,|k\rangle = |k - 1 \pmod{N}\rangle$

$$\hat{\mathrm{A}} = \mathcal{M}(c) = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$ — Sylvester's "shift matrix"

- Energy basis: $\mathcal{B}_\varepsilon = \left\{\left|\widetilde{0}\right\rangle, \left|\widetilde{1}\right\rangle, \ldots, \left|\widetilde{N-1}\right\rangle\right\}$ is basis of eigenvectors of $\hat{\mathrm{A}}$
- Diagonal form of $\hat{\mathrm{A}}$: $\hat{\mathrm{B}}\left|\widetilde{\ell}\right\rangle = \omega^\ell \left|\widetilde{\ell}\right\rangle$

$$\hat{\mathrm{B}} = F\hat{\mathrm{A}}F^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \omega & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega^{N-1} \end{pmatrix}$$ — Sylvester's "clock matrix"

$\omega = \mathrm{e}^{2\pi\mathbf{i}/N}$ is Nth primitive (ground) root of unity

$F = \frac{1}{\sqrt{N}}\left(\omega^{-k\ell}\right)$ is Fourier transform matrix

Matrices $\hat{\mathrm{A}}$ and $\hat{\mathrm{B}}$ were introduced by J.J. Sylvester in 1880s.

Symplectic structures in descriptions of evolution

- Classical mechanics: $2n$-dimensional phase space splits into pairs of conjugate variables q and p ("generalized positions and momenta").

  Poisson bracket relation: $\{q, p\} = 1$
- Standard QM: q and p are Hermitian operators.

  Heisenberg's canonical commutation relation: $[q, p] = \mathbf{i}\hbar$

  Impossible in finite-dimensional space! $\implies$ is a non-fundamental approximation

  Stone-von Neumann theorem: CCR is unique (up to unitary equivalence) in infinite-dimensional separable Hilbert space
- Finite dimension. Weyl's exponential form (Hermitian q and p are replaced with unitary Q and P) of canonical commutation relation:

  $$QP = \omega PQ, \quad \omega = \mathrm{e}^{2\pi\mathbf{i}/N} \text{ is } N\text{th ground root of unity}$$

  Uniqueness: pair $\{Q, P\}$ is unitary equivalent to the pair $\left\{\hat{\mathrm{A}}, \hat{\mathrm{B}}\right\}$ of Sylvester's "shift and clock" matrices.

Finite-dimensionality $\implies$ finiteness:

evolution is permutations of elements of a finite set

Projective representations and central extensions

- Projective (ray) unitary representation of a group G:

$$R(g)\,R(h) = \alpha(g,h)\,R(gh)$$

2-cocycles: $\alpha(g,h) \in H^2(G,\mathcal{F}^*)$
satisfy to associativity consequence

$$\alpha(f,g)\,\alpha(fg,h) = \alpha(f,gh)\,\alpha(g,h)$$

and normalization

$$\alpha(g,\mathbf{1}) = \alpha(\mathbf{1},g) = 1$$

Remark: projective representation of $\mathbb{Z}_N$ is ordinary:

$$R(c)^N = \beta\,\mathbb{1} \xrightarrow{R(c)\to\beta^{-1/N}R(c)} R(c)^N = \mathbb{1}$$

- Alternatively
Ordinary representations of central extension of G:
the set $\widetilde{G} = G \times \mathcal{F}^*$ with group operation

$$(g,a)\,(h,b) = (gh, ab\,\alpha(g,h))$$

Examples of central extensions of abelian groups

- $A^n \times A^n$ ($A = \mathbb{R}$ or $\mathbb{Z}$ or $\mathbb{Z}_{p^m}$) $\longrightarrow$ Heisenberg group H_{2n+1}

$$h \cdot h' = \begin{pmatrix} 1 & \mathbf{a} & c \\ \mathbf{0} & \mathbb{1}_n & \mathbf{b} \\ 0 & \mathbf{0} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \mathbf{a}' & c' \\ \mathbf{0} & \mathbb{1}_n & \mathbf{b}' \\ 0 & \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{a}+\mathbf{a}' & c + c' + \mathbf{a} \cdot \mathbf{b}' \\ \mathbf{0} & \mathbb{1}_n & \mathbf{b}+\mathbf{b}' \\ 0 & \mathbf{0} & 1 \end{pmatrix}$$

- Minimal example:

Klein four-group $V = \langle a, b \,|\, a^2 = b^2 = (ab)^2 = \mathbf{1} \rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$

$\longrightarrow$ quaternion group $Q_8 = \{1, \mathbf{i}, \mathbf{j}, \mathbf{k}, -1, -\mathbf{i}, -\mathbf{j}, -\mathbf{k}\}$

$$\mathbf{1} \to \{1, -1\} \to Q_8 \to V \to \mathbf{1}$$

Commutation relation: $\hat{\mathrm{A}}\hat{\mathrm{B}} = \omega \hat{\mathrm{B}}\hat{\mathrm{A}}, \quad \omega = e^{2\pi \mathbf{i}/2} = -1$

$$\hat{\mathrm{A}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \hat{\mathrm{B}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; \hat{\mathrm{A}}\hat{\mathrm{B}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Pauli matrices: $\sigma_x = \hat{\mathrm{A}}, \quad \sigma_y = \mathbf{i}\hat{\mathrm{A}}\hat{\mathrm{B}}, \quad \sigma_z = \hat{\mathrm{B}}$

Quantum complementarity as origin of symplectic conjugation

N. Bohr about quantum complementary properties: equally real but mutually exclusive.

- Mutually unbiased bases (MUB) in $\mathcal{H}_N$

$$\left|\langle a_k \,|\, b_\ell\rangle\right|^2 = \frac{1}{N} \iff \operatorname{tr}\left(A^i B^j\right) = \begin{cases} N, & \text{if } i = j = 0; \\ 0, & \text{otherwise.} \end{cases}$$

  represent complementary pairs
  - $N = p^m \implies$ maximal sets of $N+1$ MUB: composite systems of m p-dimensional subsystems
    example $N + 1 = 2 + 1$ complementary set: $\sigma_x, \ \sigma_z, \ \sigma_x\sigma_z$
    more generally, for $N = p$: $\hat{\mathrm{A}}, \hat{\mathrm{B}}, \hat{\mathrm{A}}\hat{\mathrm{B}}, \hat{\mathrm{A}}\hat{\mathrm{B}}^2, \cdots, \hat{\mathrm{A}}\hat{\mathrm{B}}^{N-1}$
  - for $N = 6, 10, 12, \ldots$ not a single example of a maximal set is known
- Ontic $\mathcal{B}_o = \{|k\rangle\}$ and energy $\mathcal{B}_\varepsilon = \left\{\left|\widetilde{\ell}\right\rangle\right\}$ bases are MUB:

$$\mathcal{B}_\varepsilon = F\mathcal{B}_o \implies \left|\widetilde{\ell}\right\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} |k\rangle\, \omega^{-k\ell}$$

$$\implies \left\langle \widetilde{\ell} \,|\, k \right\rangle = \frac{1}{\sqrt{N}}\omega^{k\ell} \implies \left|\left\langle \widetilde{\ell} \,|\, k \right\rangle\right|^2 = \frac{1}{N}$$

Projective representations of a general finite abelian group

- Group: $G \simeq \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \cdots \times \mathbb{Z}_{N_n}$
  - Generators: $\{c_j\}: \ \langle c_j \rangle \simeq \mathbb{Z}_{N_j}$
  - Relations: $c_j c_k = c_k c_j, \ c_j^{N_j} = \mathbf{1}$
- Projective representations:

$$R(g)\, R(h) = \alpha(g,h)\, R(gh)\,, \quad \alpha(g,h) \in \mathcal{F}^* \subset \mathbb{C}, \quad g,h \in G$$

- Generalized Clifford algebras:
  - Generators: $\{e_j \mid j = 1, 2, \ldots, n\}$
  - Relations: $e_j e_k = \omega_{jk} e_k e_j, \ e_j^{N_j} = \mathbb{1}, \ \omega_{jk}^{N_j} = \omega_{jk}^{N_k} = 1$

    $\omega_{jk} = \omega_{kj}^{-1} = \mathrm{e}^{2\pi \mathbf{i} s_{jk}/N_{jk}}, \ N_{jk} = \gcd(N_j, N_k)\,, \ s_{jk} \in \mathbb{Z}$

    a little unification:

    $\{N_{jk}\} \to \widetilde{N} = \mathrm{lcm}(\{N_{jk}\})\,, \ s_{jk} \to t_{jk} \in \mathbb{Z}$

    $\implies \omega_{jk} = \mathrm{e}^{2\pi \mathbf{i} t_{jk}/\widetilde{N}}\,, \ t_{kj} = -t_{jk}$

Any GCA is specified by integer $\widetilde{N}$ and antisymmetric integer matrix

$$T = \begin{pmatrix} 0 & t_{12} & \cdots & t_{1n} \\ -t_{12} & 0 & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -t_{1n} & -t_{2n} & \cdots & 0 \end{pmatrix}$$