Riemann Hypothesis Property for The Convergents of a Continued Fraction Expansion

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Abstract. We show that the denominators and numerators of convergents to a continued fraction both satisfy a Riemann Hypothesis property, meaning that their zeros lie in a perpendicular line in a complex plane.

1. Continued fraction representation

We study the function

$$\mathfrak{D}(w) = -w + \cfrac{1^2}{-w + \cfrac{2^2}{-w + \cfrac{3^2}{-w + \cfrac{4^2}{-w + \ddots}}}} = -w + \mathbf{K}_{r=1}^{\infty}(\frac{r^2}{-w}) \tag{1}$$

first formally, without verifying any convergence property. As usual, the finite initial segments $\mathfrak{D}_m(w)$ of (1) are called *convergents* ([5], [4]). For example, the first convergents are

$$\mathfrak{D}_0(w) = -w, \quad \mathfrak{D}_1(w) = -w + \frac{1}{-w} = \frac{w^2+1}{-w}, \quad \mathfrak{D}_2(w) = -w + \cfrac{1}{-w + \cfrac{4}{-w}} = \frac{-w^3 - 5w}{w^2+4}.$$

It is plain to see that each $\mathfrak{D}_n(w)$ is a rational, and denoting

$$\mathfrak{D}_n(w) = \frac{P_n(w)}{Q_n(w)}$$

the theory of continued fractions provides ([5], [4]) the recurrence relations

$$P_n(w) = -wP_{n-1}(w) + n^2 P_{n-2} \quad \text{and} \quad Q_n(w) = -wQ_{n-1}(w) + n^2 Q_{n-2}, \tag{2}$$

and the initial conditions $P_0(w) = -w$, $Q_0(1) = 1$, $P_1(w) = w^2 + 1$, $Q_1(w) = -w$ can be read from the representations of the first convergents. It is also customary to define $P_{-1}(w) = 1$ and $Q_{-1}(w) = 0$ to make the recursions (2) valid already for $n \geq 1$. By initial conditions and recursions (2) it is obvious that both $P_n(w)$ and $Q_n(w)$ are in $\mathbb{Z}[w]$ for each $n \geq 0$.

2. Determinant formulas

Lemma 1. *Let notations be as above and*

$$\hat{P}_n(w) = \begin{vmatrix} -w & 1 & 0 & 0 & \cdots & 0 & 0 \\ -3 & -w & 2 & 0 & \cdots & 0 & 0 \\ 0 & -4 & -w & 3 & \cdots & 0 & 0 \\ 0 & 0 & -5 & -w & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & n-1 & 0 \\ 0 & 0 & 0 & \cdots & -(n+1) & -w & n \\ 0 & 0 & 0 & \cdots & & -1 & -w \end{vmatrix} \qquad (3)$$

and

$$\hat{Q}_n(w) = \begin{vmatrix} -w & 2 & 0 & 0 & \cdots & 0 \\ -2 & -w & 3 & 0 & \cdots & 0 \\ 0 & -3 & -w & 4 & \cdots & 0 \\ 0 & 0 & -4 & -w & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & n \\ 0 & 0 & 0 & \cdots & -n & -w \end{vmatrix}. \qquad (4)$$

Then $\hat{P}_n(w) = P_n(w)$ and $\hat{Q}_n(w) = Q_n(w)$ for all $n \geq 1$. Notice that (3) and (6) are determinants of $(n+1)\times(n+1)$ and $n \times n$-matrices, respectively, and that the last low subdiagonal element of $\hat{P}_n(w)$ is -1 by purpose.

Proof. We prove the claim for $P_n(w)$ first. For the initial values we have $P_0(w) = -w = \det(-w) = \hat{P}_0(w)$ (the determinant of 1×1-matrix), and $P_1(w) = w^2+1 = \begin{vmatrix} -w & 1 \\ -1 & -w \end{vmatrix}$.

The recurrence for $\hat{P}_n(w)$ may not be self-evident, and hence we illustrate it here for the case $n = 4$:

$$\begin{aligned} P_4(w) &= \begin{vmatrix} -w & 1 & 0 & 0 & 0 \\ -3 & -w & 2 & 0 & 0 \\ 0 & -4 & -w & 3 & 0 \\ 0 & 0 & -5 & -w & 4 \\ 0 & 0 & 0 & -1 & -w \end{vmatrix} = \begin{vmatrix} -w & 1 & 0 & 0 & 0 \\ -3 & -w & 2 & 0 & 0 \\ 0 & -4 & -w & 3 & 0 \\ 0 & -0 & -1 & -w & 4 \\ 0 & -0 & -w & -1 & -w \end{vmatrix} \\ &= -w \begin{vmatrix} -w & 1 & 0 & 0 \\ -3 & -w & 2 & 0 \\ 0 & -4 & -w & 3 \\ 0 & 0 & -1 & -w \end{vmatrix} - 4 \begin{vmatrix} -w & 1 & 0 & 0 \\ -3 & -w & 2 & 0 \\ 0 & -4 & -w & 3 \\ 0 & 0 & -w & -1 \end{vmatrix} \\ &= -wP_3(w) - 4 \begin{vmatrix} -w & 1 & 0 & 0 \\ -3 & -w & 2 & 0 \\ 0 & -1 & -w & 3 \\ 0 & -1 & -w & -1 \end{vmatrix} = -wP_3(w) - 4 \begin{vmatrix} -w & 1 & 0 & 0 \\ -3 & -w & 2 & 0 \\ 0 & -1 & -w & 3 \\ 0 & 0 & 0 & -4 \end{vmatrix} \\ &= -wP_3(w) + 4^2 P_2(w) \end{aligned}$$

In the first line, the last column of the determinant is added to the 3rd last column, and then the Laplace expansion along the last column is applied. After this, the last column of the latter 4×4 determinant is added to the 3rd last column, and then the second-last row is added to the last one, with multiplier -1. The latest stage is the Laplace expansion along to the last row.

It is obvious that this procedure generalizes to $\hat{P}_n(w) = -w\hat{P}_{n-1}(w) + n^2\hat{P}_{n-2}(w)$ for each $n > 2$. Now that the initial conditions and the recurrence formula are same for $\hat{P}_n(w)$ and $P_n(w)$, the claim follows.

In the same way, $\hat{Q}_1(w) = -w = Q_1(w)$, $\hat{Q}_2(w) = \begin{vmatrix} -w & 2 \\ -2 & -w \end{vmatrix} = w^2 + 4 = Q_2(w)$. As a determinant of a tridiagonal matrix, $\hat{(Q)}_n(w)$ satisfies the recurrence relation $\hat{Q}_n(w) = -w\hat{Q}_{n-1}(w) + n^2\hat{Q}_{n-2}(w)$. □

Remark 1. *The numerator sequence $P_n(w)$ is equal, up to constant multipliers, to Kratwchouk polynomials ([3]). On the other hand, the denominator sequence $Q_n(w)$ is related to Meixner polynomials.*

Theorem 1. *Polynomials $P_n(w)$ and $Q_n(w)$ satisfy the RH-property (see [3]), namely, all their zeros lie in the line $\mathrm{Re}(z) = 0$.*

Proof. This follows directly from the Jacobi theorem, which states that the tridiagonal matrices of form (3) and (6) are similar to skew-symmetric matrices (for details, see [3]). □

3. Convergence questions

So far we did not consider the convergence of (1). Here we can apply the following theorem:

Theorem 2 (Van Vleck). *Let $\epsilon > 0$. Continued fraction $\mathbf{K}_{r=1}^{\infty}(\frac{1}{b_r})$ (see (1) for the notation), where $-\frac{\pi}{2} + \epsilon < \arg(b_r) < \frac{\pi}{2} - \epsilon$ converges (to a finite value) if and only if the series*

$$\sum_{r=1}^{\infty} |b_r| \tag{5}$$

diverges.

We can apply the theorem by rewriting (1) in an equivalent form

$$\mathfrak{D}(w) = -w + \cfrac{1}{-w/1 + \cfrac{1}{-w/4 + \cfrac{1}{-4w/9 + \cfrac{1}{-9w/64 + \cfrac{1}{-64w/225 + \ddots}}}}}, \tag{6}$$

which, using the notation in (1) can be written as $-w + \mathbf{K}_{r=1}^{\infty}(\frac{1}{-\xi_r w})$, where $\xi_r = (\frac{(r-1)!!}{r!!})^2$.

If $r = 2k$ is even, we can estimate

$$\frac{(r-1)!!}{r!!} = \frac{2k!}{2^k k!} : 2^k k! = \frac{1}{2^{2k}}\binom{2k}{k} \sim \frac{1}{\sqrt{\pi k}},$$

and the estimate is similar for odd r. Therefore, $\xi_r = \Theta(\frac{1}{r})$ and the series (5) with $b_r = \xi_r w$ obviously diverges whenever $w \neq 0$. It follows that the continued fraction expansion (1) is convergent for all $w \neq 0$, provided $\arg b_r \neq \pm\frac{\pi}{2}$.

References

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