On the Integrability of the Polynomial Case of a Liénard-type Equation

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Abstract. The paper investigates the connection between the global integrability of an autonomous two-dimensional polynomial ODE system and its local integrability near stationary points using the example of the polynomial case of a Lenar-type equation. We presented the equation in the form of a dynamical system and parametrized it. The conditions for local integrability near stationary points are written out and the values of the parameters under which these conditions are satisfied are found. It is established that for certain values of the parameters obtained in this way, the system actually turns out to be integrable. Thus, we can speak of a heuristic approach that allows one to determine the cases of ODE integrability.

Introduction

We use an approach based on local analysis. It uses the resonant normal form computed near stationary points [1]. In [2], a method was proposed for finding parameter values for which the dynamical system is locally integrable at all stationary points simultaneously. The main idea is that in the domain of integrability in the phase space, a necessary condition is local integrability at every point of this domain. But at regular points, local integrability already takes place, so local integrability is also necessary at singular points, and at all such points of the domain under consideration.

Note that for the global integrability of an autonomous planar system, it suffices to have one global integral of motion. From its expression, one can obtain a solution of the system in quadratures; therefore, integrability implies the solvability of the system.

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Problem

We will check our method on the example of the Liénard-like equation

$$\ddot{x} = f(x)\dot{x} + g(x), \tag{1}$$

Here we assume that $f(x)$ and $g(x)$ are polynomials. Usually, the Linard equation assumes that $f(x)$ is an even function and $g(x)$ is an odd function [3] . We do not assume a certain parity for them, so we are talking about the Linard-type equation.

Equation (1) is equivalent to the dynamical system

$$\begin{aligned} &\dot{x} = y, \\ &\dot{y} = (a_0 + a_1 x)\, y + b_1 x + b_2 x^2 + b_3 x^3, \end{aligned} \tag{2}$$

here x and y are functions in time and parameters a_0, a_1, b_1, b_2, b_3 are real.

The problem is to construct the first integrals of system (2).

Method

Note that for the global integrability of an autonomous planar system, it suffices to have one global integral of motion. From its expression, one can obtain a solution of the system in quadratures; therefore, integrability implies the solvability of the system. The main task of the method under discussion is to find conditions on the parameters of the system under which the system is locally integrable near its stationary points. Local integrability means the presence of a sufficient number (one for an autonomous flat system) of local integrals at each point of the region under study, including the corresponding stationary points. Local integrals may be different for each point of this region of the phase space, but for the existence of a global integral, local integrals must exist for the desired values of the parameters at all fixed points. This is a necessary condition. In papers [1] the algebraic condition of local integrability is written out. This is the so-called **A** condition. This condition is satisfied at all regular points, but it is nontrivial at stationary points.

First, we look for sets of parameters under which the condition **A** is satisfied at the fixed point of the system at the origin (2). We solve the corresponding systems of algebraic equations with respect to the parameters a_0, a_1, b_1, b_2, b_3 and check the integrability at other stationary points for each found set of parameters. "Good" sets of parameters are good candidates for the existence of a single function for all points - the first integral. These integrals are sought by the method described below.

Conditions of the Integrability

The condition **A** is some infinite sequence of polynomial equations with respect to the coefficients of the system. Each of the stationary points has its own system of equations. But the normal form has a non-trivial form only in the resonant case.

This means that we can only use our method if the eigenvalues of the linear part of the system (2) refer as integers. We restrict our study to this case for now. A possible condition for these eigenvalues to be related as $1 : M$ and have opposite signs (the resonance case) is the relation

$$a_0 - \sqrt{a_0^2 + 4b_1} = -M\left(a_0 + \sqrt{a_0^2 + 4b_1}\right).$$

We choose the "resonance" restriction on parameters in the form

$$b_1 = \frac{a_0^2 M}{(M-1)^2}. \tag{3}$$

From the A condition, we constructed three equations for the system parameters a_0, a_1, b_2, b_3 for the $(1:2), (1:3)$ and $(1:4)$ resonances, i.e. for $M = 2, 3, 4$. Here is the first of three equations for $M = 2$ as an example

$$a_0{}^3\left(2a_1{}^3 - 29a_1b_3\right) + a_0{}^2b_2\left(26a_1{}^2 + 43b_3\right) + 13a_0a_1b_2{}^2 - 11b_2{}^3 = 0.$$

Results

For the case with resonance $M = 2$ the solutions of the corresponding algebraic system calculated by the MATHEMATICA-11 system are

$$\begin{array}{ll} 1) & \{a_0 \to 0, b_2 \to 0\},\\ 2) & \{b_2 \to -a_0a_1, b_3 \to 0\},\\ 3) & \{b_2 \to -4a_0a_1/7, b_3 \to -6/49\, a_1{}^2\},\\ 4) & \{b_2 \to -a_0a_1/3, b_3 \to -a_1{}^2/9\},\\ 5) & \{b_2 \to 3a_0a_1, b_3 \to a_1{}^2\},\\ 6) & \{a_1 \to 0, b_2 \to 0, b_3 \to 0\}. \end{array} \tag{4}$$

Here $b_1 \to 2a_0^2$ for $M = 2$. At these sets of parameters we checked the integrability condition at other stationary points of (2).

The autonomous system of the second order(2) can be rewrite as the first order non-autunomous equation

$$\begin{array}{l} dy(x)/dx = [(a_0 + a_1x)\, y(x) + b_1x + b_2x^2 + b_3x^3]/y(x)\\ \quad \text{or}\\ dx(y)/dy = x(y)/[(a_0 + a_1x(y))\, y + b_1x(y) + b_2x(y)^2 + b_3x(y)^3]. \end{array} \tag{5}$$

After this rewrite, we tried to solve such equations with each of the parameter sets (4) using the MATHEMATIC-11 solver. We have found solutions for sets 2), 4), 5) and 6) in the implicit form of $F(y(x), x, C) = 0$. We then expressed the constant C as a function in $x, y(x)$ and replaced these variables with $x(t)$ and $y(t)$. Thus, we obtain integrals of motion. The resulting integrals can be verified by direct calculation of the time derivative along the system.

The integrable cases in (4) correspond to the equations

$$\begin{array}{ll} 2) & \ddot{x} = (a_0 + a_1 x)\,\dot{x} + 2{a_0}^2 x - a_0 a_1 x^2, \\ 4) & \ddot{x} = (a_0 + a_1 x)\,\dot{x} + 2{a_0}^2 x - \frac{1}{3} a_0 a_1 x^2 - \frac{1}{9}{a_1}^2 x^3, \\ 5) & \ddot{x} = (a_0 + a_1 x)\,\dot{x} + 2{a_0}^2 x + 3 a_0 a_1 x^2 + {a_1}^2 x^3. \end{array} \tag{6}$$

We returned here from the systems of equations to the equations of the second order. For equation 2) the first integral is

$$\frac{(a_1 x(t) - 2a_0)\sinh\left(\frac{1}{2}R(x(t), y(t))\right) + a_0 R(x(t), y(t))\cosh\left(\frac{1}{2}R(x(t), y(t))\right)}{(a_1 x(t) - 2a_0)\cosh\left(\frac{1}{2}R(x(t), y(t))\right) + a_0 R(x(t), y(t))\sinh\left(\frac{1}{2}R(x(t), y(t))\right)},$$

where

$$R(x(t), y(t)) = \sqrt{\frac{a_1(x(t)(a_1 x(t) - 2a_0) - 2y(t))}{{a_0}^2}}.$$

We have done the above steps for $M = 2, 3$ resonances with the same results. The coefficient b_1 for x in (6) is fixed everywhere, since $b_1 = 2a_0^2$ for $M = 2$, but for other M it will be different.

Note also that case 4) in (6) (6) is exactly a special case of equation 4 from section 2.2.3-2 of [4] with parameters $a_0 \to b, a_1 \to 3a, c \to 2b^2$. But the fact that cases 2) and 5) are integrable is a new result, at least for this book.

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