On the Integrability of the Polynomial Case of a Liénard-type Equation

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Integrability

For an ODE system the integrals of motion satisfy the relation

$$\frac{d\ I_i(x_1,\ldots,x_n,t)}{d\ t}=0 \quad \text{along the system} \quad \frac{d\ x_i}{d\ t}=\phi_i(x_1,\ldots,x_n,t).$$

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- Integrability is a very important property of the system. In particular, if a system is integrable then it is solvable by quadrature.
- The knowledge of the integrals is important at the investigation of a phase portrait, for the creation of symplectic integration schemes e.t.c.

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The Liénard equation

- Let f and g be two continuously differentiable functions on $\mathbb{R}$, where g is an odd and f is an even function. Then the second-order ordinary differential equation of the form

$$\ddot{x} = f(x)\dot{x} + g(x)$$

  is called the Liénard equation.
- **Theorem** Liénard equation has a unique and stable limit cycle surrounding the origin if it satisfies the following additional properties...
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Model system

- We look at polynomial form of the Liénard-like equation

$$\ddot{x} = (a_0 + a_1 x)\,\dot{x} + b_1 x + b_2 x^2 + b_3 x^3 = 0,$$

here x is a function in time and parameters $a_0, a_1, b_1, b_2, b_3 \in \mathbb{R}$.

- We rewrite it as autonomous polynomial dynamical system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= (a_0 + a_1 x)\,y + b_1 x + b_2 x^2 + b_3 x^3, \end{aligned} \tag{1}$$

- We look for the parameters at which the first integral exists.

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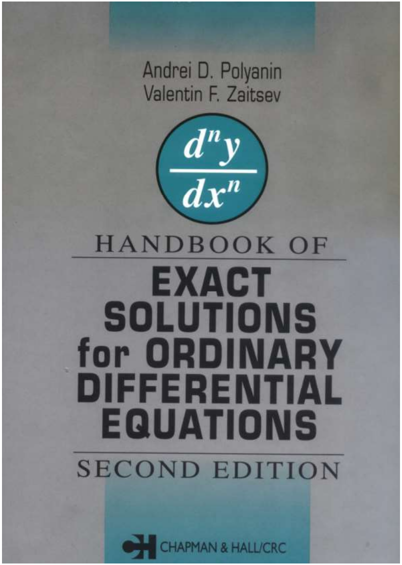
Polyanin's and Zaitsev's book (2003)



2.2.3-2. Solvable equations and their solutions.

1. $y''_{xx} + y + ay^3 = 0.$

Duffing equation. This is a special case of equation 2.9.1.1 with $f(y) = -y - ay^3$.

1°. Solution:

$$x = \pm \int \left(C_1 - y^2 - \tfrac{1}{2}ay^4\right)^{-1/2} dy + C_2.$$

The period of oscillations with amplitude C is expressed in terms of the complete elliptic integral of the first kind:

$$T = \frac{4}{\sqrt{1+aC^2}} K\left(\frac{aC^2}{2+2aC^2}\right), \quad \text{where} \quad K(m) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-m\sin^2 t}}.$$

2°. The asymptotic solution, as $a \to 0$, has the form:

$$y = \widetilde{C}_1 \cos[(1 + \tfrac{3}{8}a\widetilde{C}_1^2)x + \widetilde{C}_2] + \tfrac{1}{32}a\widetilde{C}_1^3 \cos[3(1 + \tfrac{3}{8}a\widetilde{C}_1^2)x + 3\widetilde{C}_2] + O(a^2),$$

where $\widetilde{C}_1$ and $\widetilde{C}_2$ are arbitrary constants. The corresponding asymptotics for the period of oscillations with amplitude C is described by the formula: $T = 2\pi(1 - \frac{3}{8}aC^2) + O(a^2)$.

2. $y''_{xx} + ayy'_x + by^3 + cy = 0.$

The transformation $w(z) = y'_x$, $z = -\frac{1}{2}ay^2$ leads to an Abel equation of the form 1.3.1.2: $ww'_z - w = -\dfrac{2b}{a^2}z + \dfrac{c}{a}$.

3. $y''_{xx} = (ay + 3b)y'_x + cy^3 - aby^2 - 2b^2y.$

The substitution $w(y) = y'_x$ leads to an Abel equation of the form 1.3.3.1:

$$ww'_y = (ay + 3b)w + cy^3 - aby^2 - 2b^2y.$$

4. $y''_{xx} = (3ay + b)y'_x - a^2y^3 - aby^2 + cy.$

The substitution $w(y) = y'_x$ leads to an Abel equation of the form 1.3.3.2:

$$ww'_y = (3ay + b)w - a^2y^3 - aby^2 + cy.$$

5. $2y''_{xx} = (7ay + 5b)y'_x - 3a^2y^3 - 2cy^2 - 3b^2y.$

The substitution $w(y) = y'_x$ leads to an Abel equation of the form 1.3.3.3:

$$2ww'_y = (7ay + 5b)w - 3a^2y^3 - 2cy^2 - 3b^2y.$$

Main Assumption

- We assume that the existence of a global integral in some region of the phase space requires the condition of local integrability in a neighborhood of each point of the region under study, at least for the polynomial systems. That is, local integrability at every point is required.
- Local integrals can be represented by different functions at different points, but they must exist for the global integrability.
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Scheme

Any system is locally integrable in neighborhoods of regular points of the phase space and locally integrable in all cases of stationary points, except for resonant ones. Therefore, our approach requires the study of the local integrability at all stationary points of the region under study, where resonance takes place,

- We fix the parameters of the system under study in such a way that it has a resonance at some fixed point.
- We create and solve the necessary conditions of the local integrability in the parametric space in the resonant stationary point. We get sets of parameter values that are candidates for a search of global integrability.
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The resonance normal form

- The resonance normal form was introduced by Poincaré for the investigation of systems of nonlinear ordinary differential equations. It is based on the maximal simplification of the right-hand sides of these equations by invertible transformations.
- The normal form approach was developed in works of G.D. Birkhoff, T.M. Cherry, A. Deprit, F.G. Gustavson, C.L. Siegel, J. Moser, A.D. Bruno et.al. This technique is based on the Local Analysis method by Prof. Bruno.

Multi-index notation

Let's suppose that we treat the polynomial system and rewrite this n-dimension system in the terms

$$\dot{x}_i = \lambda_i x_i + x_i \sum_{\mathbf{q}\in\mathbb{N}_i} f_{i,\mathbf{q}} \mathbf{y}^{\mathbf{q}}, \quad i = 1, \ldots, n\,, \tag{2}$$

where we use the multi-index notation

$$\mathbf{x}^{\mathbf{q}} \equiv \prod_{j=1}^{n} {x_j}^{q_j},$$

with the power exponent vector $\mathbf{q} = (q_1, \ldots, q_n)$
Here the sets:

$$\mathbb{N}_i = \{\mathbf{q} \in \mathbb{Z}^n : q_i \geq -1 \text{ and } q_j \geq 0\,,\ \text{if } j \neq i\,, \quad j = 1, \ldots, n\}\,,$$

because the factor y_i has been moved out of the sum in (2).

Normal form

The normalization is done with a near-identity transformation:

$$x_i \;=\; z_i + z_i \sum_{\mathbf{q}\in\mathbb{N}_i} h_{i,\mathbf{q}}\mathbf{z}^{\mathbf{q}}, \quad i = 1, \ldots, n \tag{3}$$

after which we have system (2) in the normal form:

$$\dot{z}_i = \lambda_i z_i + z_i \sum_{\substack{\langle \mathbf{q}, \mathbf{L}\rangle = 0 \\ \mathbf{q}\in\mathbb{N}_i}} g_{i,\mathbf{q}}\mathbf{z}^{\mathbf{q}}, \quad i = 1, \ldots, n\,. \tag{4}$$

Resonance terms

- The important difference between (2) and (4) is a restriction on the range of the summation, which is defined by the equation:

$$\langle \mathbf{q}, \mathbf{L} \rangle = \sum_{j=1}^{n} q_j \lambda_j = 0 . \qquad (5)$$

- **L** here is the vector of the eigenvalues of matrix of linear part of the system (2). The **q**-terms in the normal form (4) are terms, for which (5) is valid. They are called resonance terms.

Note, if the eigenvalues are not comparable then condition (5) is never valid at any components of the vector **q**, because they are integer. For example such situation takes place if $\lambda_1 = 1, \lambda_2 = \sqrt{2}$. In that case the normal form (4) will be a linear system.

Calculation of the Normal form

The h and g coefficients in (3) and (4) are found by using the recurrence formula:

$$g_{i,\mathbf{q}} + \langle \mathbf{q}, \mathbf{L} \rangle \cdot h_{i,\mathbf{q}} = -\sum_{j=1}^{n} \sum_{\substack{\mathbf{p}+\mathbf{r}=\mathbf{q} \\ \mathbf{p},\mathbf{r} \in \bigcup_i \mathbb{N}_i \\ \mathbf{q} \in \mathbb{N}_i}} (p_j + \delta_{ij}) \cdot h_{i,\mathbf{p}} \cdot g_{j,\mathbf{r}} + \tilde{\Phi}_{i,\mathbf{q}} \,, \quad (6)$$

For this calculation, we wrote two programs, in LISP and the high-level language of the MATHEMATICA system.

Conditions A and ω

There are two conditions:

- Condition A. In the normal form (4)

$$g_j(Z) = \lambda_j a(Z) + \bar{\lambda}_j b(Z), \quad j = 1, \ldots, n,$$

  where $a(Z)$ and $b(Z)$ are some formal power series.
- Condition ω (on small divisors). It is fulfilled for almost all vectors **L**. At least it is satisfied at rational eigenvalues.
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Near a stationary point the condition **A:**

- Ensures convergence.
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Condition **A**

- The condition **A** is an infinite system of algebraic equations on the system parameters.
- It can be calculated by the CA program till some finite order. It will be the necessary condition of the local integrability as a finite system of algebraic equations in the system parameters. Sometimes it is solvable.

Resonant Case of the System

- We fix b_1 in system (1) in such way that the system has the resonance $1 : M$ at the origin point

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= (a_0 + a_1 x)\, y + \frac{a_0^2 M}{(M-1)^2} x + b_2 x^2 + b_3 x^3. \end{aligned} \tag{7}$$

- The matrix of the linear part of this system is

$$\begin{pmatrix} 0 & 1 \\ \frac{{a_0}^2 M}{(M-1)^2} & a_0 \end{pmatrix}, \quad M \neq 1,$$

with eigenvalues

$$\left\{ \frac{-a_0}{M-1}, \frac{a_0 M}{M-1} \right\}.$$

- Their ratio is -M. This is a resonant case. We started from $M = 2$.

Conditions of the Local Integrability

- We calculated the normal form for resonances (1:2), (1:3), and (1:4) till the ninth, twelfth and fifteenth orders.
- For each case, we have obtained the algebraic systems in a parametric space with three equations. Here is the first of three equations for $M = 2$, for example

$$a_0{}^3\left(2a_1{}^3 - 29a_1b_3\right) + a_0{}^2b_2\left(26a_1{}^2 + 43b_3\right)$$
$$+13a_0a_1b_2{}^2 - 11b_2{}^3 = 0.$$

- These systems are huge, but the MATHEMATICA-11 solver finds all solutions in a few seconds.
- We publish in this report mainly the case $M = 2$. But other resonances demonstrate the same picture.

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Solutions of the Conditions

The MATHEMATICA solver found 6 solutions

$$\begin{array}{ll} a) & \{a_0 \to 0, b_2 \to 0\}, \\ b) & \{b_2 \to -a_0 a_1, b_3 \to 0\}, \\ c) & \{b_2 \to -4a_0 a_1/7, b_3 \to -6/49\, a_1{}^2\}, \\ d) & \{b_2 \to -a_0 a_1/3, b_3 \to -a_1{}^2/9\}, \\ e) & \{b_2 \to 3a_0 a_1, b_3 \to a_1{}^2\}, \\ f) & \{a_1 \to 0, b_2 \to 0, b_3 \to 0\}. \end{array} \tag{8}$$

And we put the resonant condition $b_1 \to 2a_0^2$ for $M = 2$.

At each of these sets of parameters we checked the integrability conditions at other stationary points of the system.

The corresponding equation are

$$
\begin{array}{ll}
a) & \ddot{x} = a_1 x\,\dot{x} + b_3 x^3, \\
b)^* & \ddot{x} = (a_0 + a_1 x)\,\dot{x} + 2{a_0}^2 x - a_0 a_1 x^2, \\
c) & \ddot{x} = (a_0 + a_1 x)\,\dot{x} + 2{a_0}^2 x - \frac{4}{7} a_0 a_1 x^2 - \frac{1}{49} 6{a_1}^2 x^3, \\
d)^* & \ddot{x} = (a_0 + a_1 x)\,\dot{x} + 2{a_0}^2 x - \frac{1}{3} a_0 a_1 x^2 - \frac{1}{9} {a_1}^2 x^3, \\
e)^* & \ddot{x} = (a_0 + a_1 x)\,\dot{x} + 2{a_0}^2 x + 3 a_0 a_1 x^2 + {a_1}^2 x^3, \\
f)^* & \ddot{x} = a_0 \dot{x} + 2a_0^2 x.
\end{array}
\tag{9}
$$

We returned here from the systems of equations to the equations of the second order for comparison with the book results.

The asterisk (*) marks the integrable case, discussed below.

Solvable Equations from the Book

$$1)\ddot{y} = -\dot{y} - a\,y^3,$$

$$2)\ddot{y} = -a\,y\,\dot{y} - b\,y^3 - c\,y,$$ a) is partial case of 2)

$$3)\ddot{y} = (a\,y + 3\,b)\,\dot{y} - 2\,b^2\,y - a\,b\,y^2 + c^3,$$ a) is partial case of 3)

$$4)\ddot{y} = (3\,a\,y + b)\,\dot{y} + c\,y - a\,b\,y^2 - a^2 y^3,$$ d) & f) are partial case of 4)

$$5)\ddot{y} = \left(\tfrac{7}{2}\,a\,y + \tfrac{5}{2}\,b\right)\,\dot{y} - \tfrac{3}{2}\,b^2 y - c\,y^2 - \tfrac{3}{2}\,a^2\,y^3.$$

Note, that equations 1) and 5) do not intersect with (9).

Cases b), c) and e) are not reflected in the book.

Moreover, as we show below, cases b), d), e) and f) are integrable. So, integrability of equations b) and e) is a new result.

Higher resonances

The integrable cases take places for higher resonances also. For example, case d) has the form

$$\begin{aligned}
d) \quad & \ddot{x} = (a_0 + a_1 x)\,\dot{x} + 2{a_0}^2 x \quad -\tfrac{1}{3} a_0 a_1 x^2 - \tfrac{1}{9} {a_1}^2 x^3, \quad \text{for } M = 2,\\
d) \quad & \ddot{x} = (a_0 + a_1 x)\,\dot{x} + \tfrac{3{a_0}^2}{4} x \quad -\tfrac{1}{3} a_0 a_1 x^2 - \tfrac{1}{9} {a_1}^2 x^3, \quad \text{for } M = 3,\\
d) \quad & \ddot{x} = (a_0 + a_1 x)\,\dot{x} + \tfrac{4{a_0}^2}{9} x \quad -\tfrac{1}{3} a_0 a_1 x^2 - \tfrac{1}{9} {a_1}^2 x^3, \quad \text{for } M = 4.
\end{aligned}$$

Hypothesis

$$\ddot{x} = (a_0 + a_1 x)\,\dot{x} + c\,x - \frac{1}{3} a_0 a_1 x^2 - \frac{1}{9} {a_1}^2 x^3, \quad \text{i.e. b1 = c ?} \qquad (10)$$

First Integrals of Motion

The autonomous system of the second order can be rewrote as the first order non-autunomous equation. Let

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y).$$

We divided the left and right sides of the system equations into each other. In result we have the first-order differential equations for $x(y)$ or $y(x)$

$$\frac{dx}{dy} = \frac{P(x, y)}{Q(x, y)} \quad \text{or} \quad \frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)}.$$

Then we solved them by the MATHEMATICA solver and sometimes got solutions $y(x)$ (or $x(y)$). After that we calculated the integrals from these solutions by extracting the integration constants $I(x, y) = const$.

- By this method we integrated 4 from 6 cases in (8). They correspond to the equations b), d), e) and f).
- In cases a) and c) we cannot obtain solutions in this way, but this does not prove their non-integrability. For example a) is the partial case of the integrable equation 2) from the book. Later we hope to try the Darboux method.

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For case b) the first integral is

$$I(x,y) = \frac{(a_1 x - 2a_0)\sinh\left(\frac{1}{2}R(x,y)\right) + a_0 R(x,y)\cosh\left(\frac{1}{2}R(x,y)\right)}{(a_1 x - 2a_0)\cosh\left(\frac{1}{2}R(x,y)\right) + a_0 R(x,y)\sinh\left(\frac{1}{2}R(x,y)\right)},$$

where

$$R(x,y) = \sqrt{\frac{a_1(x(a_1 x - 2a_0) - 2y)}{{a_0}^2}}.$$

For f) case the first integral is

$$I(x,y) = (2a_0 x - y)^2(a_0 x + y).$$

But the integrals for d) and e) are defined implicitly by transcendental equations.

Conclusions

- Integrable cases b), d), e) and f) in (9) of equation (1) found by the algorithmic method.
- Case d) is just a special case of equation 4) from section 2.2.3-2 of [PolyaninZaitsev 2003] with parameters $a_0 \to b$, $a_1 \to 3a$, $c \to 2b^2$. But with the hypothesis (10) above this is exact that equation from the book.
- The fact that cases b) and e) are integrable is a new result, at least from the point of view of the book.
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- Case d) is just a special case of equation 4) from section 2.2.3-2 of [PolyaninZaitsev 2003] with parameters $a_0 \to b$, $a_1 \to 3a$, $c \to 2b^2$. But with the hypothesis (10) above this is exact that equation from the book.
- The fact that cases b) and e) are integrable is a new result, at least from the point of view of the book.
- We considered here only resonances at the origin of the system (1). Resonances near other stationary points can give the more results. The same can be said about the study of higher resonances.

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Many thanks for your attention

Example

For example, we have had treated the system

$$dx/dt = -y^3 - b\,x^3 y + a_0\,x^5 + a_1\,x^2 y^2,$$
$$dy/dt = (1/b)\,x^2 y^2 + x^5 + b_0\,x^4 y + b_1\,x\,y^3,$$

with five arbitrary real parameters $b \neq 0, a_1, a_2, b_1, b_2$. Using the Power Geometry method, we brought the system to a non-degenerate form.

With our technique, we found seven two-dimensional conditions at which the system above is integrable

$$
\begin{array}{ll}
1)\ b_1 = 0, & a_0 = 0, a_1 = -b_0\, b, \ \ b^2 \neq 2/3; \\
2)\ b_1 = -2\, a_1, & a_0 = a_1\, b, b_0 = b_1\, b, \ b^2 \neq 2/3; \\
3)\ b_1 = 3/2\, a_1, & a_0 = a_1\, b, b_0 = b_1\, b, \ b^2 \neq 2/3; \\
4)\ b_1 = 8/3\, a_1, & a_0 = a_1\, b, b_0 = b_1\, b, \ b^2 \neq 2/3; \\
5)\ b_1 = 3/2\, a_1, & a_0 = (2b_0 + b\,(3a_1 - 2b_1))/3, \ b = \sqrt{2/3}; \\
6)\ b_1 = 6\, a_1 + 2\sqrt{6} b_0, & a_0 = (2b_0 + b\,(3a_1 - 2b_1))/3, \ b = \sqrt{2/3}; \\
7)\ b_1 = -2/3\, a_1, & 20a_0 + 2\sqrt{6}a_1 + 4b_0 + 3\sqrt{6}b_1 = 0, \\
 & 3a_0 - 2b_0 \neq b\,(3a_1 - 2b_1), \qquad b = \sqrt{2/3}.
\end{array}
$$

For each of these conditions, we found the first integral of motion.