Computation of Hamiltonian high order normal form

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Abstract. The procedure of deriving homological equations of arbitrary order, which solutions are used in iterative procedure of normalization of a Hamiltonian in a neighborhood of an equilibrium position, is considered. A formula for a homological equation of arbitrary order used in the method of normalization by means of the Lie series is proposed. The normalization procedure is applied to Hamiltonian of the Hill problem written in scaled regular variables. The resulting normal form of the Hill problem can be used to find domains of analyticity of the normalizing transformation.

Introduction

Normal form (NF) of a system of ordinary differential equations (ODE) computed near an invariant manifold (stationary point, periodic solution or invariant torus) is rather powerful technique for investigation of local dynamics of the phase flow in the vicinity of this invariant structure. Even though the NF is a formal object it can be used for searching first integrals of the system, families of periodic solutions, for studying integrability, stability and bifurcations. The special properties of Hamilton systems require specific algorithms for computation their NF. The goal of the presented work is to provide a procedure for constructing so called homological equation of any order, which is used in the procedure of so called invariant Hamiltonian normalization .

1. Hamiltonian normal form

We consider an analytic Hamiltonian system

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \mathbf{y}}, \quad \dot{\mathbf{y}} = -\frac{\partial H}{\partial \mathbf{x}} \tag{1}$$

with n degrees of freedom near its stationary point $\mathbf{x} = \mathbf{y} = 0$.

The Hamiltonian function $H(\mathbf{x},\mathbf{y})$ is expanded into convergent power series $H(\mathbf{x},\mathbf{y}) = \sum H_{\mathbf{pq}}\mathbf{x}^{\mathbf{p}}\mathbf{y}^{\mathbf{q}}$ with constant coefficients $H_{\mathbf{pq}}$, $\mathbf{p},\mathbf{q} \geq 0$, $|\mathbf{p}|+|\mathbf{q}| \geq 2$.

Canonical transformations of coordinates $\mathbf{x},\mathbf{y}$

$$\mathbf{x} = \mathbf{f}(\mathbf{u},\mathbf{v}), \quad \mathbf{y} = \mathbf{g}(\mathbf{u},\mathbf{v}), \tag{2}$$

preserve the Hamiltonian character of the initial system (1).

Denoting by $\mathbf{z} = (\mathbf{x},\mathbf{y}) \in \mathbb{R}^{2n}$ the phase vector one can write the linear part of the system (1) in the form

$$\dot{\mathbf{z}} = B\mathbf{z}, \quad B = \frac{1}{2}\, J \operatorname{Hess} H|_{\mathbf{z}=0}\,, \quad J = \begin{pmatrix} 0^n & E^n \\ -E^n & 0^n \end{pmatrix},$$

where J is symplectic unit matrix, E^n is identity matrix and $\operatorname{Hess} H$ is Hessian of function H. Let $\lambda_1,\ldots,\lambda_{2n}$ be eigenvalues of the matrix B, which can be reordered in such a way that $\lambda_{j+n} = -\lambda_j$, $j = 1,\ldots,n$. Denote by $\boldsymbol{\lambda} = (\lambda_1,\ldots,\lambda_n)^{\mathrm{T}}$.

There exists [1, § 12, Theorem 12] a canonical formal transformation (2) in the form of power series, which reduces the initial system (1) into its *normal form* for the case of semi-simple eigenvalues

$$\dot{\mathbf{u}} = \frac{\partial h}{\partial \mathbf{v}}, \dot{\mathbf{v}} = -\frac{\partial h}{\partial \mathbf{u}},$$

defined by the normalized Hamiltonian

$$h(\mathbf{u},\mathbf{v}) = \sum_{j=1}^{n} \lambda_j u_j v_j + \sum h_{\mathbf{pq}}\mathbf{u}^{\mathbf{p}}\mathbf{v}^{\mathbf{q}},$$

containing only resonant terms $h_{\mathbf{pq}}\mathbf{u}^{\mathbf{p}}\mathbf{v}^{\mathbf{q}}$ with

$$\langle \mathbf{p}-\mathbf{q}, \boldsymbol{\lambda} \rangle = 0.$$

Here $0 \leq \mathbf{p},\mathbf{q} \in \mathbb{Z}^n$, $|\mathbf{p}|+|\mathbf{q}| \geq 2$ and $h_{\mathbf{pq}}$ are constant coefficients.

2. Invariant normalization method and its application

Here we describe normalization procedure.

- The real Hamiltonian $H(\mathbf{x},\mathbf{y})$ is written in the complex form $H(\mathbf{z},\bar{\mathbf{z}})$.
- The method of invariant normalization is applied to $H(\mathbf{z},\bar{\mathbf{z}})$ up to the definite order and we get it NF $h(\mathbf{Z},\bar{\mathbf{Z}})$, which contains only resonant terms.
- The obtained complex NF $h(\mathbf{Z},\bar{\mathbf{Z}})$ can be transformed into the real NF $h(\mathbf{X},\mathbf{Y})$.

Here we consider a Hamiltonian system, which stationary point (SP) coincides with the origin. Applying scaling $\mathbf{x} \to \varepsilon\mathbf{x}$, $\mathbf{y} \to \varepsilon\mathbf{y}$ and $t \to \varepsilon^2 t$ one can write it in the form of power series in ε: $H(\mathbf{x},\mathbf{y}) = H^0 + F = H^0 + \sum_{j=1}^{\infty} \varepsilon^j H_j(\mathbf{x},\mathbf{y})$, where H^0 is quadratic (unperturbed) form and H_j is a homogeneous form of order $j+2$. We are looking for the NF of the original Hamiltonian H as a power series $h(\mathbf{z},\bar{\mathbf{z}}) = h^0 + f = h^0 + \sum_{j=1}^{\infty} \varepsilon^j h_j(\mathbf{z},\bar{\mathbf{z}})$, where $h^0 = \sum_{j=1} \lambda_j z_j \bar{z}_j$ and homogeneous forms h_j, $j > 0$, contain only resonant terms $h_{\mathbf{pq}}\mathbf{z}^{\mathbf{p}}\bar{\mathbf{z}}^{\mathbf{q}}$, $|\mathbf{p}|+|\mathbf{q}| = j+2$, such that $\langle \boldsymbol{\lambda}, \mathbf{p} - \mathbf{q} \rangle$. Transformation from the initial Hamiltonian H to its NF h is provided by Lie generator G having form of a power series of ε: $G = \sum_{j=1} \varepsilon^j G_j$: $h = H + \sum_{j=1}^{\infty} \frac{1}{j!} H * G^j$. Lie generator G produces a near identical transformation, so we have $h^0 = H^0$ and then

$$f = h^0 * G + M, \quad M = F + \sum_{j=1} \frac{1}{j!} H * G^j. \tag{3}$$

Solution of (3) can be obtained be the *method of invariant normalization*, proposed by V.F. Zhuravlev [2, 3]. This method can be considered as subsequent averaging of functions M_j along the unperturbed solutions $\mathbf{z}(t, \mathbf{Z}, \bar{\mathbf{Z}})$ obtained from the unperturbed system with Hamiltonian H^0. It can be applied for the case of nonzero eigenvalues.

According to it the homological equations can be rewritten in the form

$$\frac{df_j}{dt} = 0, \quad M_j = f_j - \frac{dG_j}{dt}, \quad j = 1, 2, \ldots \tag{4}$$

Substituting the solutions $\mathbf{z}(t, \mathbf{Z}, \bar{\mathbf{Z}})$, $\bar{\mathbf{z}}(t, \mathbf{Z}, \bar{\mathbf{Z}})$ to the unperturbed system into the function M_j one gets function $m_j(t, \mathbf{Z}, \bar{\mathbf{Z}}) = M_j(t, \mathbf{Z}, \bar{\mathbf{Z}})$ and getting the following quadrature

$$\int_0^t m_j(t, \mathbf{Z}, \bar{\mathbf{Z}}) dt = t f_j(\mathbf{Z}, \bar{\mathbf{Z}}) + G_j(\mathbf{Z}, \bar{\mathbf{Z}}) + g(t). \tag{5}$$

Hence, on each step of the normalization procedure the next term of the NF f_j equals the coefficient at t, and the Lie generator term G_j equals the time-independent term in (5).

It is possible to reduce approximately in 4 times the number of terms in functions M_j, $j = 2, 3, \cdots$. From the first equation of (3) for each $j = 2, 3, \ldots$ one can get that $h^0 * G_j = f_j - M_j$. Let us introduce the following notations:

$$f_j^+ \equiv F_j + f_j, f_j^- \equiv F_j - f_j, H * G^k_{j_1 \cdots j_k} = H * G^{k-1}_{j_1 \cdots j_{k-1}} * G_{j_k}.$$

Statement 1. *For $j > 2$ function M_j is constructed in a such way:*

- *Term F_j is taking and sum $\frac{1}{2} \sum_{k=1}^{j-1} f_k^+ * G_{j-k}$ is adding to it.*
- *For each k not greater than $[j/2]$ we compute the set $\mu_{2k+1}(j)$ of all permutations of any partition $\nu_{2k+1}(j)$, i.e. the set $\mu_{2k+1}(j)$ contains the tuple of $2k+1$ indices which sum is equal to j. For each such tuple $(i_1, \ldots, i_{2k+1})$ of indices one has to compute all the Poisson brackets of form $f_{i_1}^- * G^{2k}_{i_2 \cdots i_{2k+1}}$.*
- *The sum all the computed above Poisson brackets is multiplied by the coefficient α_{2k}. These coefficients are well known Bernoulli numbers B_{2k} divided by factorial $(2k)!$: $\alpha_{2k} = \frac{B_{2k}}{(2k)!}$. They can be computed with the help of generating function $\mathfrak{g}(\varepsilon) = \frac{\varepsilon}{2} + \frac{\varepsilon}{e^\varepsilon - 1} - 1$.*

- *The final formula for M_j can be written as follows*

$$M_j = F_j + \frac{1}{2}\sum_{k=1}^{j-1} f_k^+ * G_{j-k} + \sum_{k=1}^{[j/2]} \alpha_{2k} \sum_{(i_1,\dots,i_{2k+1})\in\mu^j_{2k+1}} f_{i_1}^- * G^{2k}_{i_2\cdots i_{2k+1}}.$$

It is evident that high order normalization of the Hamiltonian H is only possible with computer algebra systems. For example, such software [3, Ch. 7] was developed in CAS `Wolfram Mathematica`. The author implemented the described above algorithm in CAS `Maplesoft Maple`. Nevertheless, this invariant normalization method can be implemented in other open source CAS. For example, in `SageMath`, which essentially uses the `SymPy` symbolic computation package, or `Maxima`.

The method of invariant normalization was applied to the well know planar circular Hill problem, which Hamiltonian written in scaled regularized variables has polynomial form. The NF h in the vicinity of the origin was computed up to the 20-th order. This NF can be used for asymptotic integration of the Hill's problem equations of motion and for studying so called domains of analyticity [4].

References

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