On the topology and singularity theory of functions on two-dimensional sphere

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Introduction

Let be a locally trivial bundle of smooth compact manifolds $F \hookrightarrow W \to M$ and smooth general function $f : W \to \mathbb{R}$ in total space. The purpose of this work is to relate the topological characteristics of the bundle, which serve as an obstruction to its triviality, with the types of degeneracies of the critical points of the restriction of the function to fibers. If the bundle is trivial, $W = M \times F$, then there is such a function f, whose restriction on each fiber is Morse, for example, we can take fixed Morse function $g\colon F \to \mathbb{R}$ and set $f = g \circ p_2$ where $p_2 : W \to F$ is a natural projection to the second factor. If the bundle $W \to M$ is nontrivial then we can expect the existence of fibers where restriction of f will have degeneracy. This connection may be used in two directions: on the one hand, by studying the singularities for a given function, one can try to build topological obstructions to the triviality of the bundle (i.e. characteristic classes). On the other hand, the nontriviality of the bundle should imply the necessity of the existence of singularities.

In the case when the bundle layer is a circle, the theory describing the correspondence between the characteristic classes of the bundle and the singularities of the function on its layers is constructed in [1]. In this paper, we take the first step in extending this theory to the case when the bundle fiber is a two-dimensional oriented sphere, and the singularities of the global minimum are considered as singularities. We give a construction of the classifying space for this case. It has homotopy type $BSO(3)$, and hence its cohomology groups are known. On the other hand, the space itself is divided into strata that correspond to one-to-one types of degenerations of the global minimum of functions on the sphere. We give a classification of the singularities of the global minimum of functions on a sphere of small codimension, and give a description of the homotopy type of most strata of small codimension as well as two of infinite series of strata. This information is used to construct a spectral sequence converging to the cohomology of the classifying space.

The main result is description of strata of small codimensions (up to codimension 4) and two infinite series of strata (1)$[1^n]$, 2) $[2k-1]$; 3) $[2k-1,1]$ which means respectively 1) strata of functions with n points of global minima each of them is a singularity of A_1 type; 2) strata of functions with one point of global minima that has a A_{2k-1} type, 3) strata of functions with two points of global minima one of them of type A_{2k-1} and the second of type A_1).

Список литературы

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