Notes on Obstacles to Dimensionality Reduction

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Abstract. We consider the arrangement of vertices of the unit multidimensional cube and affine subspace as well as their orthographic projections onto coordinate hyperplanes. Upper and lower bounds on the subspace dimension are given under which some orthographic projection always preserves the incidence relation between the subspace and cube vertices. The proved upper bound is equal to the integer part of half the dimension of the ambient space.

Introduction

Let us consider the recognition problem whether there is a $(0,1)$-solution to a system of linear equations over a field K, where $\mathrm{char}(K) \neq 2$. From a geometric point of view, we consider the recognition problem whether a given affine subspace passes through a vertex of the multidimensional unit cube. Over the ring of integers, the problem is known as the multiple subset sum problem. It is NP-complete. Nevertheless, there are known many heuristic methods [1, 2, 3, 4].

Let us consider the n-dimensional affine space with a fixed system of Cartesian coordinates. The vertices of the unit n-dimensional cube are points with coordinates equal to either zero or one. These vertices are called $(0,1)$-points for short. As usual, a cube in the plane is called a square.

Let us formulate the second problem. Given an affine subspace L that is not incident to any $(0,1)$-point. Does there exist a projection onto a low-dimensional coordinate subspace that forgets some coordinates so that the image of the subspace L is also not incident to any $(0,1)$-point? It is important that the image of the cube is again a low-dimensional cube. The problem is closely related to pseudo-Boolean programming and various generalizations of the knapsack problem. Of course, such dimensionality reduction reduces the computational complexity. The imposed conditions ensure the correctness of reducing the original problem to a problem with a smaller number of variables. But in the worst case, any hyperplane cannot serve as an image of a subspace L under the considered restrictions.

In Euclidean space, for dimensionality reduction by means of some projection, one can use probabilistic algorithms based on the Johnson–Lindenstrauss lemma [5, 6]. However, this approach is not applicable over an arbitrary field.

For $n > m$, a projection $K^n \to K^m$ is so-called *orthographic* when the projection forgets some coordinates. The term was historically used to denote orthogonal projections from three-dimensional space onto a plane over reals. For an affine space over an arbitrary field K, the notion of orthogonality has no meaning. Nevertheless, using a fixed coordinate system, it is possible to define a special class of projections onto coordinate subspaces. We hope this term will not lead to misunderstanding.

Results

Theorem 1. *Given a positive integer s. There is an s-dimensional affine subspace $L \subset K^{2s+1}$ such that L does not pass through any $(0,1)$-point, but under the orthographic projection onto a coordinate hyperplane, the image of the plane L passes through some $(0,1)$-point.*

Proof. Let us denote by A the point with coordinates $(1/2, 0, \dots, 0)$, where all but one of the coordinates are zeros. For $1 \le k \le s$, the point $A^{(2k)}$ has coordinates $(0, \dots, -1, 1, \dots)$, where $A^{(2k)}_{2k} = -1$, $A^{(2k)}_{2k+1} = 1$, and other coordinates are zeros. The point $A^{(2k+1)}$ has coordinates $(1, \dots, 1, -1, \dots)$, where $A^{(2k+1)}_1 = 1$, $A^{(2k+1)}_{2k} = 1$, $A^{(2k+1)}_{2k+1} = -1$, and other coordinates are zeros. All points A, $A^{(2k)}$, and $A^{(2k+1)}$ belong to an affine subspace L, which is defined by a system of linear equations: $1 - 2x_1 + x_2 + \dots + x_{2k} + \dots + x_{2s} = 0$ and other s equations $x_{2k} + x_{2k+1} = 0$, where $1 \le k \le s$.

The inequality $\dim L \le s$ holds because these equations are linearly independent. On the other hand, for all $1 \le k \le s$, three points A, $A^{(2k)}$, and $A^{(2k+1)}$ belongs to a straight line. All these lines intersect each other at the point A. The inequality $\dim L \ge s$ holds because the affine hull of these s straight lines is s-dimensional. Thus, $\dim L = s$.

Under the orthographic projection onto a coordinate hyperplane, the image of L passes through a $(0,1)$-point. This point is an image of some point from the set A, $A^{(2)}$, …, $A^{(2s+1)}$. Let us check that no $(0,1)$-point belongs to L. If all even coordinates vanish $x_{2k} = 0$, then $x_1 = 1/2$ in accordance with the first equation. Otherwise, for some k, both equalities $x_{2k} = 1$ and $x_{2k+1} = -1$ hold. □

Example 1. Let us consider three points in a three-dimensional affine space with coordinates $(0, 1, 1/2)$, $(1, 2, 0)$, and $(-1, 0, 1)$, respectively. These points belong to the same straight line L, which can be given by a system of two equations $x_2 = x_1 + 1$ and $x_3 = (-x_1 + 1)/2$. But under the orthographic projection onto any coordinate plane, the image of this set of three points contains some $(0,1)$-point.

Remark 1. The characteristic of the field K does not equal two because we use division by two.

Theorem 2. *Given a positive integer s. Over any infinite field K, there is an s-dimensional affine subspace $L \subset K^{2s}$ such that L does not pass through any $(0,1)$-point, but under the orthographic projection onto a coordinate hyperplane, the image of the plane L passes through some $(0,1)$-point.*

Remark 2. In Theorem 2, the field K is infinite because the proof uses the Schwartz–Zippel lemma. In fact, the same theorem holds over the field having exactly three elements. It is unknown whether it holds over larger finite fields.

Example 2. Let us consider the plane in the four-dimensional affine space that is defined by the system of two equations $x_3 = x_1+x_2+1$ and $x_4 = (-x_1+x_2+1)/2$. A straightforward check shows that this plane does not pass through any $(0,1)$-point. However, this plane passes through the points $(-1,0,0,1)$, $(0,-1,0,0)$, $(0,1,2,1)$, $(0,0,1,1/2)$, each of which has exactly one coordinate different from both zero and one. Therefore, under the orthographic projection onto any coordinate hyperplane, the image of this plane is incident to some $(0,1)$-point.

Theorem 3. *For all straight lines $L \subset K^n$, where $n \geq 4$, if L is not incident to any $(0,1)$-point, then there is an orthographic projection onto some coordinate hyperplane such that the image of the line L is also not incident to any $(0,1)$-point.*

Theorem 4. *For all planes $L \subset K^n$, where $n \geq 6$, if L is not incident to any $(0,1)$-point, then there is an orthographic projection onto some coordinate hyperplane such that the image of the plane L is also not incident to any $(0,1)$-point.*

Remark 3. Over a finite field K, one can obtain a lower bound for the dimension of an affine subspace $L \subset K^n$ such that L does not pass through any $(0,1)$-point, but under the orthographic projection onto any coordinate hyperplane, the image of L is incident to some $(0,1)$-point. If the field K consists of q elements, then s-dimensional subspace has q^s elements. So, the bound is $s \geq \log_q n$.

Conclusion

Our results illustrate the high computational complexity of pseudo-Boolean programming because the reduction in the dimension of the ambient space by means of projection meets an obstacle in the worst case. Moreover, in small dimensions, we know the exact bound for the dimension of the subspace for which the discussed obstacle to dimensionality reduction exists. However, such an obstacle arises only for special arrangements of the affine subspace. In the general case and over an infinite field, there is an orthographic projection so that the image of the subspace is a hyperplane in a space of lower dimension and the image is not incident to any $(0,1)$-point. Of course, there may be many such projections. Moreover, for different projections, the computational complexity of checking whether no $(0,1)$-points is incident to the resulting subspace may be greater or less.

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