Notes on Obstacles to Dimensionality Reduction

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Let us consider the recognition problem whether there is a $(0,1)$-solution to a system of linear equations over a field K, where char$(K) \neq 2$. From a geometric point of view, we consider the recognition problem whether a given affine subspace passes through a vertex of the multidimensional unit cube.
Over the ring of integers, the problem is known as the multiple subset sum problem. It is NP-complete.
Over a finite field of odd characteristic, the problem is NP-complete too.

The problem is closely related to pseudo-Boolean programming and various generalizations of the knapsack problem.

We consider dimensionality reduction.

Let us consider the n-dimensional affine space with a fixed system of Cartesian coordinates. The vertices of the unit n-dimensional cube are points with coordinates equal to either zero or one. These vertices are called $(0, 1)$-points for short.

For $n > m$, a projection $K^n \to K^m$ is so-called *orthographic* when the projection forgets some coordinates.

It is important that the image of the unit cube is again a low-dimensional unit cube.

The term was historically used to denote orthogonal projections from three-dimensional space onto a plane over reals. For an affine space over an arbitrary field K, the notion of orthogonality has no meaning. Nevertheless, using a fixed coordinate system, it is possible to define a special class of projections onto coordinate subspaces.

The current problem. Given an affine subspace L that is not incident to any $(0, 1)$-point. Does there exist a projection onto a low-dimensional coordinate subspace that forgets some coordinates so that the image of the subspace L is also not incident to any $(0, 1)$-point?

Theorem. *Given a positive integer s. There is an s-dimensional affine subspace $L \subset K^{2s+1}$ such that L does not pass through any $(0,1)$-point, but under the orthographic projection onto any coordinate hyperplane, the image of L passes through some $(0,1)$-point.*

Example. Let us consider three points in a three-dimensional affine space with coordinates $(0,1,1/2)$, $(1,2,0)$, and $(-1,0,1)$, respectively. These points belong to the same straight line L, which can be given by a system of two equations $x_2 = x_1 + 1$ and $x_3 = (-x_1 + 1)/2$. But under the orthographic projection onto any coordinate plane, the image of this set of three points contains some $(0,1)$-point.

Theorem. *Given a positive integer s. Over any infinite field K, there is an s-dimensional affine subspace $L \subset K^{2s}$ such that L does not pass through any $(0,1)$-point, but under the orthographic projection onto any coordinate hyperplane, the image of L passes through some $(0,1)$-point.*

Remark. In this theorem, the field K is infinite because the proof uses the Schwartz–Zippel lemma.
In fact, the same theorem holds over the field having exactly three elements.
It is unknown whether it holds over larger finite fields.

Example. Let us consider the plane in the four-dimensional affine space that is defined by the system of two equations $x_3 = x_1 + x_2 + 1$ and $x_4 = (-x_1 + x_2 + 1)/2$. A straightforward check shows that this plane does not pass through any $(0,1)$-point. However, this plane passes through the points $(-1,0,0,1)$, $(0,-1,0,0)$, $(0,1,2,1)$, $(0,0,1,1/2)$, each of which has exactly one coordinate different from both zero and one. Therefore, under the orthographic projection onto any coordinate hyperplane, the image of this plane is incident to some $(0,1)$-point.

The rank of a matrix A is related to the dimensionality of the affine hull L of all points corresponding to columns of the matrix. If L passes through the origin, then $\operatorname{rank}(A) = \dim(L)$, else $\operatorname{rank}(A) = \dim(L) + 1$.

Theorem. *For every odd n, there is a $n \times n$ matrix A over the field K such that every entry outside the leading diagonal belongs to the set $\{0, 1\}$, every diagonal entry is neither 0 nor 1, no $(0, 1)$-point belongs to the affine hull of all points corresponding to columns of the matrix A, and the equality $\operatorname{rank}(A) = \lceil n/2 \rceil$ holds.*

Proof. Let us consider the $n \times n$ matrix

$$A = \begin{pmatrix} 1/2 & 0 & 1 & 0 & 1 & \cdots & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -1 \end{pmatrix}.$$

Theorem. *Given a $n \times n$ matrix A over the field K, where every entry outside the leading diagonal belongs to the set $\{0, 1\}$, but every diagonal entry is neither 0 nor 1. The rank of the matrix A is at least $n/2$.*

Theorem. *Given an even n and $n \times n$ matrix A over the field K, where every entry outside the leading diagonal belongs to the set $\{0, 1\}$, but every diagonal entry is neither 0 nor 1. If no $(0, 1)$-point belongs to the affine hull of all points corresponding to columns of the matrix A, then the rank of the matrix A is at least $(n/2) + 1$.*

Theorem. *For subspaces $L \subset K^n$, if $\dim L < \lfloor n/2 \rfloor$ and L does not pass through any $(0, 1)$-point, then there is an orthographic projection onto some coordinate hyperplane such that the image of L does not pass through any $(0, 1)$-point.*

Theorem. *If a straight line L intersects each of some three straight lines containing pairs of adjacent $(0,1)$-points, but L does not pass through any $(0,1)$-point, then L lies in the affine hull of a $(0,1)$-point and three adjacent $(0,1)$-points.*

Proof. One can assume that the straight line L intersects the first coordinate axis at the point A with coordinates $(a, 0, \ldots, 0)$, where all coordinates except the first one are equal to zero and $a \notin \{0, 1\}$. There is some $k \geq 2$ such that the straight line L passes through a point W for which all coordinates except the k-th one belong to the set $\{0, 1\}$.

The straight line L consists of points $(1 - t)A + tW$, where t denotes a parameter. If some two coordinates of the point W except the first one equal to one, then these coordinates of any third point on the line L are different from both zero and one. However, according to the condition, there is a third point on the line L for which exactly one coordinate is different from both zero and one. So, the point W can have at most three coordinates different from zero including the first one. Therefore, the straight line L lies in a coordinate subspace of dimensionality at most three.

Let us consider five straight lines $G^{(1)}$, ..., $G^{(5)}$ in the 5-dimensional space such that each the line contains a pair of adjacent $(0,1)$-points. If a plane $L \subset K^5$ does not pass through any $(0,1)$-point, but L intersects five straight lines $G^{(1)}$, ..., $G^{(5)}$, then there is a straight line in L that intersects three of these straight lines. So, every discussed obstacle to dimensionality reduction has a hierarchical structure.

Theorem. *If a plane L intersects each of some five straight lines $G^{(1)}$, ..., $G^{(5)}$ containing pairs of adjacent $(0,1)$-points, but L does not pass through any $(0,1)$-point, then L lies in the affine hull of a $(0,1)$-point and five adjacent $(0,1)$-points.*

Let us consider the field $\mathbb{F}_3$ of residues modulo 3.

Theorem. *Over the field $\mathbb{F}_3$, if there is no $(0,1)$-solution to a linear equation*

$$x_n = a_0 + a_1 x_1 + \cdots + a_{n-1} x_{n-1},$$

then $a_0 = 2$ and other coefficients vanish: $a_1 = 0$, ..., $a_{n-1} = 0$.

Proof. If the free term a_0 equals 0 or 1, then a $(0,1)$-solution exists, e.g., $x_1 = 0$, ..., $x_{n-1} = 0$, and $x_n = a_0$.
Else $a_0 = 2$.
The equation $x_n = 2$ has no $(0,1)$-solution.
Otherwise, one can assume $a_0 = 2$ and $a_1 \in \{1,2\}$.
A $(0,1)$-solution exists, for example,
$x_1 = 1$, $x_2 = 0$, ..., $x_{n-1} = 0$, and $x_n = a_1 - 1$.

Conclusion

Our results illustrate the high computational complexity of pseudo-Boolean programming because the reduction of the ambient space by means of projection meets an obstacle in the worst case. Moreover, we know the exact bound for the dimension of the subspace for which the discussed obstacle to dimensionality reduction exists. However, such an obstacle arises only for special arrangements of the affine subspace. In the general case and over an infinite field, there is an orthographic projection so that the image of the subspace is a hyperplane in a space of lower dimension and the image is not incident to any $(0, 1)$-point.
Of course, there may be many such projections. For different projections, the computational complexity of checking whether no $(0, 1)$-point is incident to the resulting subspace may be greater or less.

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