Bernstein polynomials and MacWilliams transform

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Abstract. In this report we show that the vector of coefficients of the Bernstein polynomial (in monomial basis) for a function given on the interval $[-1, 1]$ is (up to a rational multiplier) the MacWilliams transform of the vector of selected samples of this function taken with binomial weights.

Keywords. Bernstein polynomials, Krawtchouk polynomials, MacWilliams matrices, Pascal-MacWilliams pyramid, Cellular automata.

Introdution

Bernstein polynomials are apparently the first historical example of a constructive proof of Weierstrass approximation theorem. These polynomials are widely used for approximation problems alongside with other methods (such as the least-squares method) and play an important role in computer graphics, as one of the forms of analytical representation of Bézier curves. [2, p. 41]

1. Bernstein polynomials and Krawtchouk polynomials

The classical definition of Bernstein polynomials is as follows:

Definition 1.1. *Let $f(x) \in C[0, 1]$. The Bernstein polynomial $B_n(f; x)$ of degree n for the sampling vector $f_r = f(x_r) = (f(0), f(1/n), \ldots, f(1))$ on the uniform grid $x_r = r/n$, $r = 0, 1, \ldots, n$ is defined as the polynomial*

$$B_n(f; x) = \sum_{r=0}^{n} \binom{n}{r} f_r x^r (1-x)^{n-r}, \tag{1}$$

where the products $\binom{n}{r} x^r (1-x)^{n-r}$ are called Bernstein basis polynomials or Bézier polynomials.

The approximation property of these polynomials is expressed by the following theorem:

Theorem 1.1. *If $f(x)$ is a continuous function on the interval $[0,\,1]$, then as $n \to \infty$, the sequence of polynomials $B_n(f;\,x)$ converges uniformly on the interval $[0,\,1]$ to the function $f(x)$.*

For other intervals, it is necessary to change the variable. For the purposes of this paper we introduce a new variable

$$t = 2x - 1 \quad \text{i.e.} \quad x = \frac{t+1}{2}. \tag{2}$$

It is easy to see that with this new variable, the domain of fucntion f is the interval $[-1,\,1]$.

Formula (1), in accordance with (2), can be rewritten as follows

$$B_n(f;\,t) = \frac{1}{2^n}\sum_{r=0}^{n}\binom{n}{r} f_r (1+t)^r (1-t)^{n-r}, \tag{3}$$

where $f_r = f(t_r) = (f(-1),\,\ldots,\,f(1))$, $r = 0,\,\ldots,\,n$ — $n+1$-vector of samples of the function f at the points t_r.

Definition 1.2. *The coefficients of the powers of z in the polynomial $(1+z)^{n-r}(1-z)^r$ are obviously polynomials of r; they are called Krawtchouk polynomials of order n. In other words, the polynomial $(1+z)^{n-r}(1-z)^r$ is a generating function for Krawtchouk polynomials of order n* [3, ch. 5, §2]*:*

$$(1+z)^{n-r}(1-z)^r = \sum_{s=0}^{n} K_s^{(n)}(r)\, z^s. \tag{4}$$

Due to the trivial identity

$$(1+t)^r(1-t)^{n-r} = (1+t)^{n-(n-r)}(1-t)^{n-r},$$

the formula (3) can be rewritten to the form

$$B_n(f;\,t) = \frac{1}{2^n}\sum_{r=0}^{n}\binom{n}{r} f_{n-r}\sum_{s=0}^{n} K_s^{(n)}(n-r)t^s, \tag{5}$$

where $f_r = f(t_r) = f(-1),\, f(1)$, $r = 0,\,\ldots,\,n$ and since $\binom{n}{r} = \binom{n}{n-r}$, we finally obtain

$$B_n(f;\,t) = \frac{1}{2^n}\sum_{s=0}^{n}\left(\sum_{r=0}^{n}\binom{n}{r}\, f_{n-r}K_s^{(n)}(r)\right)\cdot t^s, \tag{6}$$

which represents the expansion of the Bernstein polynomial in terms of the powers of the variable t. In the next section, we will provide a closed form for the coefficients of t^s in (6) using the definition of the MacWilliams transform, widely used in algebraic coding theory.

2. Bernstein polynomials and MacWilliams transform

Definition 2.1. *A square $(n+1)\times(n+1)$-matrix M_n, where*

$$(M_n)_{ij} = K_i^{(n)}(j), \quad 0 \le i,\, j < n \tag{7}$$

is called a MacWilliams matrix (see [4, p. 4, 18]*).*

For any column vector $u = (u_0, u_1, \ldots, u_n)$ of length $(n+1)$ its MacWilliams transform of order n is defined as the product

$$\mathcal{M}_n(u) = M_n\, u.$$

From the properties of Krawtchouk polynomials ([3], [4]), one can easily get the properties of MacWilliams matrices. Here are some of these properties:

Let $C = \operatorname{diag}\left(\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}\right)$ and $\mathbf{I}$ is the identity matrix. Then the following relations hold:

1. Explicit formula: $K_s^{(n)}(r) = \sum_{l=0}^{s}(-1)^l\binom{n-r}{s-l}\binom{r}{l}$;
2. Free term of Krawtchouk polynomial: $K_r^{(n)}(0) = \binom{n}{r}$;
3. Orthogonality: $\sum_{i=0}^{n}\binom{n}{i}K_r^{(n)}(i)K_s^{(n)}(i) = 2^n\binom{n}{r}\delta_{r,\,s}$, i.e. $M_nCM_n^T = 2^nC$;
4. Involutiveness: $\sum_{i=0}^{n}K_r^{(n)}(i)K_i^{(n)}(s) = 2^n\delta_{r,\,s}$, i.e. $M_n^2 = 2^n\,\mathbf{I}$ and $M_n^{-1} = \frac{1}{2^n}M_n$
5. Reciprocity formula: $\binom{n}{r}K_s^{(n)}(r) = \binom{n}{s}K_r^{(n)}(s)$ i.e. $M_n^T = C^{-1}M_nC$

Some examples of MacWilliams matrices are shown below in 3.1.1 (see also [4])

Let ${}^{\beta}f = \left(\binom{n}{r}\cdot f_{n-r}\right)^T_{0\le r\le n}$ be the column vector of the samples of the function f with binomial weights, and let $\mathrm{T}_n(f)$ be the vector of coefficients of the Bernstein polynomial $B_n(f;\,t)$ in the basis t^s.

Then, it is easy to see that using the introduced notation, formula (6) can be written as

$$\mathrm{T}_n(f) = \frac{1}{2^n}\mathrm{M}_n\,{}^{\beta}f = \mathrm{M}_n^{-1}\,{}^{\beta}f, \tag{8}$$

which allows us to represent the Bernstein polynomial as

$$B_n(f;\,t) = \sum_{s=0}^{n}(\mathrm{T}_n(f))_s t^s. \tag{9}$$

Our previous considerations can be formulated as follows:

Proposition 2.1. *The $(n+1)$-dimensional vector of coefficients of the Bernstein polynomial $T_n(f)$ is a MacWilliams transform of the reverse vector of samples of the function f with the binomial weight, divided by 2^n.*

3. Pascal-MacWilliams pyramid

In this section, we will show that the set of MacWilliams matrices can be naturally represented as a three-dimensional pyramid, where the horizontal sections are the matrices M_n, and each such section is the algebraic sum of the shifts of the previous section, similar to what happens for the rows of Pascal's triangle, which justifies the name of this pyramid [1].

For such representation, we will use the results of [4]. We introduce the following notation used in this work.

Definition 3.1. *For any matrix A, we define its zero-padded matrix $\overline{|A}$, where the horizontal (vertical) bar stands for $\mathbf{0}$ — a row (column) of zeros:*

$$\overline{|A} = \left[\begin{array}{c|c} 0 & \mathbf{0} \\ \hline \mathbf{0} & A \end{array}\right].$$

The notation $\overline{A|}$, $\underline{|A}$, and $\underline{A|}$ have a similar meaning.

With previous notation, the construction of the Pascal-MacWilliams pyramid is given by the following theorem:

Theorem 3.1. *For the MacWilliams matrices M_n, the following recurrence relation holds (a detailed proof is given in [4, p. 7]):*

$$M_{n+1} = \left(\underline{M_n|} + \overline{M_n|} + \underline{|M_n} - \overline{|M_n}\right) \cdot \operatorname{diag}(1,\, 1/2,\, \ldots,\, 1/2,\, 1),\ n \geq 0,\ M_0 = [1]. \tag{10}$$

Example 3.1.1. *Let's list the MacWilliams matrices, computing them using formula (10):*

$$M_0 = [1];$$

$$M_1 = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix};$$

$$M_2 = \left(\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}\right) \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix};$$

$$M_3 = \left(\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 0 & -2 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 0 & -2 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 2 & 0 & -2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 0 & -2 \\ 0 & 1 & -1 & 1 \end{bmatrix}\right) \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & 3 \\ 3 & -1 & -1 & 3 \\ 1 & -1 & 1 & -1 \end{bmatrix}.$$

4. Conclusion

1. Proposition 2.1 can be reformulated as follows: Let $V_n = \mathbb{Z}_2^n$ be the so-called dyadic group of dimension n, that is, an n-dimensional vector space over the field $\mathbb{Z}_2$. Let $f : V_n \to \mathbb{R}$ be a real-valued function on this space defined by the formula:
$$\forall v \in V_n \quad f(v) = f(|v|) = f_k, \tag{11}$$
where $k = |v|$ is the Hamming weight of the vector v, $0 \le k \le n$.
   Then Proposition 2.1 together with formula (11) show that problem of finding the coefficient-list $T_n(f)$ of the Bernstein polynomial is the problem of harmonic analysis on V_n, since the Krawtchouk polynomials $K_r^{(n)}$ are themselves the Fourier transforms (or, equivalently, the Hadamard transforms) on the group V_n of characteristic functions of Hamming spheres
$$S_r = \{v \in V_n \mid |v| = r\},\ 0 \le r \le n.$$
2. Similarly to how (as is well-known) Pascal's triangle can be considered as the result of successive states of a certain one-dimensional cellular automaton, the Pascal-MacWilliams pyramid can also be interpreted as the result of the operation of a similar but two-dimensional automaton, which was presented by one of the authors of this publication in the Wolfram Library Archive in 2004: https://library.wolfram.com/infocenter/MathSource/5223/

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