Bernstein polynomials and MacWilliams transform

Nikita Gogin, Vladislav Shubin

Polynomial Computer Algebra, April 2023

Bernstein polynomials

Definition 1

Let $f(x) \in C[0, 1]$. The Bernstein polynomial $B_n(f; x)$ of degree n for the sampling vector $f_r = f(x_r) = (f(0), f(1/n), \ldots, f(1))$ is defined as the polynomial

$$B_n(f; x) = \sum_{r=0}^{n} \binom{n}{r} f_r x^r (1-x)^{n-r}. \tag{1}$$

Definition 1.1

The Bernstein basis polynomial is defined as

$$b_{n,r} = \binom{n}{r} x^r (1-x)^{n-r}.$$

Krawtchouk polynomials

Definition 2

The polynomial $(1+z)^{n-r}(1-z)^r$ is a generating function for Krawtchouk polynomials of order n:

$$(1+z)^{n-r}(1-z)^r = \sum_{s=0}^{n} K_s^{(n)}(r)\, z^s. \qquad (2)$$

implicit form for Krawtchouk polynomial of order n and degree s is

$$K_s^{(n)}(z) = \sum_{i=0}^{s} (-1)^i \binom{z}{i}\binom{n-z}{n-i}$$

With all previous definitions we can rewrite formula (1) as follows:

$$B_n(f;\ t) = \frac{1}{2^n} \sum_{s=0}^{n} \left(\sum_{r=0}^{n} \binom{n}{r} f_{n-r} K_s^{(n)}(r) \right) \cdot t^s. \qquad (3)$$

Therefore, formula (3), in its turn, can be written in matrix form:

MacWilliams transform

Definition 3

A square $(n+1) \times (n+1)$-matrix M_n, where

$$(M_n)_{ij} = K_i^{(n)}(j), \quad 0 \le i, j < n \tag{4}$$

is called a MacWilliams matrix.
For any column vector $u = (u_0, u_1, \ldots, u_n)$ of length $(n+1)$ its MacWilliams transform of order n is defined as the product

$$\mathcal{M}_n(u) = M_n\, u.$$

Bernstein polynomials and MacWilliams transform

Let ${}^{\beta}f = \left(\binom{n}{r} \cdot f_{n-r}\right)^{T}_{0 \le r \le n}$ and let $\mathrm{T}_n(f)$ be the vector of coefficients of the Bernstein polynomial $B_n(f;\, t)$.
Then

$$\mathrm{T}_n(f) = \frac{1}{2^n}\mathrm{M}_n\, {}^{\beta}f = \mathrm{M}_n^{-1}\, {}^{\beta}f, \tag{5}$$

which allows us to represent the Bernstein polynomial as

$$B_n(f;\, t) = \frac{1}{2^n}\sum_{s=0}^{n}(\mathcal{M}_n({}^{\beta}f))_s t^s. \tag{6}$$

This formula is closed form for Bernstein polynomials (in t^s basis) through MacWilliams transform.

Pascal-MacWilliams pyramid

A set of all MacWilliams matrices can be visualised as a three-dimensional pyramid with the help of the following recurrent relation:

$$M_{n+1} = \left(\underline{M_n|} + \overline{M_n|} + \underline{|M_n} - \overline{|M_n}\right) \cdot \mathrm{diag}(1,\ 1/2,\ \ldots,\ 1/2,\ 1),$$
$$n \geq 0,\ \ M_0 = [1]. \qquad (7)$$

The way of construction is similar to well-known Pascal triangle. We'll list some MacWilliams matrices and their constructions on the next slide.

Examples

$$M_0 = [1];$$

$$M_1 = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix};$$

$$M_2 = \left(\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \right)$$

$$\cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix};$$

Examples

$$M_3 = \left(\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 0 & -2 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 0 & -2 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 2 & 0 & -2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \right.$$

$$\left. - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 0 & -2 \\ 0 & 1 & -1 & 1 \end{bmatrix} \right) \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & 3 \\ 3 & -1 & -1 & 3 \\ 1 & -1 & 1 & -1 \end{bmatrix}.$$

Examples

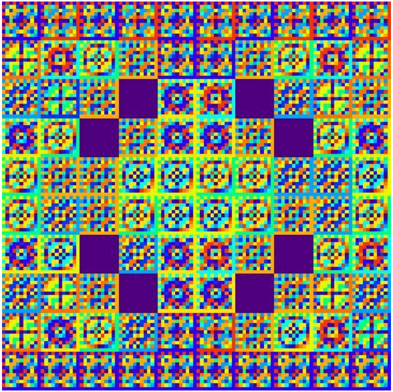

Figure: Example of MacWilliams matrix of order 109 modulo 11.

Examples

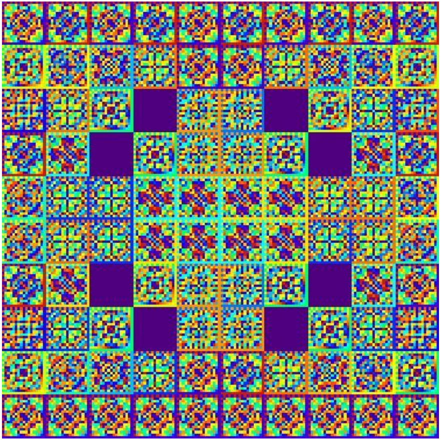

Figure: Example of MacWilliams matrix of order 169 modulo 17.

Examples

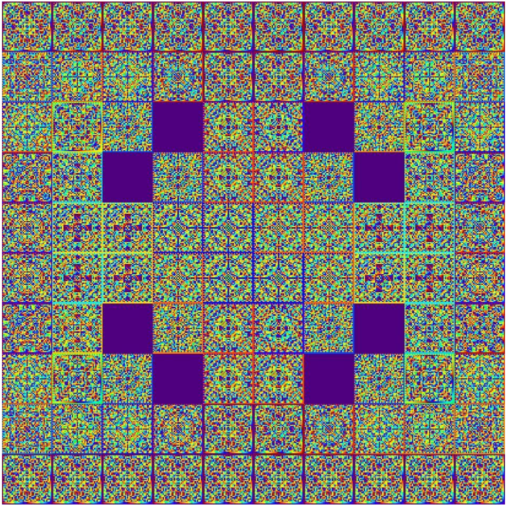

Figure: Example of MacWilliams matrix of order 409 modulo 41.

Final remark: cellular automaton

Similarly to how (as is well-known) Pascal's triangle can be considered as the result of successive states of a certain one-dimensional cellular automaton, the Pascal-MacWilliams pyramid can also be interpreted as the result of the operation of a similar but two-dimensional automaton, which was presented by one of the authors of this publication in the Wolfram Library Archive in 2004: `https://library.wolfram.com/infocenter/MathSource/5223/`.

References

Bondarenko Boris A.
Generalized Pascal triangles and pyramids.
The Fibonacci Association, 1993.

MacWilliams F.J. and Sloane N.J.A.
The theory of error correcting codes.
North-holland Publishing Company, 1977.

Gogin Nikita and Hirvensalo Mika.
Recurrent construction of MacWilliams and Chebyshev matrices.
Fundam. Informaticae, 16(1–4):93–110, 2012.