
# LEVI-CIVITA REDUCTION IN GENERAL THREE-BODY PROBLEM

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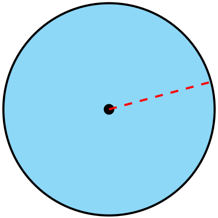
# Two body problem

## The energy integral

$$
\begin{aligned}
T - U &= \frac{\dot{\mathbf{r}}^2}{2} - \frac{1}{r} = h \\
r &\leq -\frac{2}{h}, \quad h < 0.
\end{aligned}
$$



# Two body problem

The energy integral

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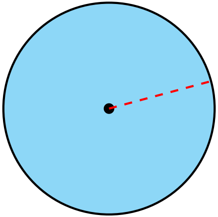


Adding the angular momentum integral

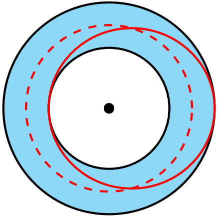
$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{J} = r^2\dot{\theta}$$

yields

$$\frac{2}{r} + h - \frac{c^2}{r^2} \geq 0$$

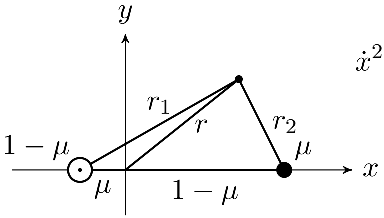
or

$$\begin{cases} r_{\mathsf{min}} \leq r \leq r_{\mathsf{max}} & \text{if } h < 0, \\ r > r_{\mathsf{min}} & \text{if } h \geq 0 \end{cases}$$



Cartesian relative or baricentric coordinate system

# CR3BP

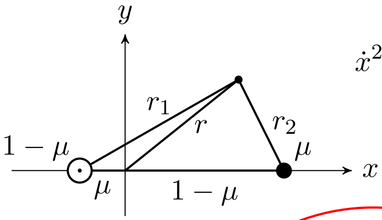



Jacobi's integral:

$$\dot{x}^2+\dot{y}^2+\dot{z}^2-(x^2+y^2)-2\left(\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right)+C=0,$$

$$\begin{aligned} r_1 &= \sqrt{(x+\mu)^2+y^2+z^2}, \\ r_2 &= \sqrt{(x+\mu-1)^2+y^2+z^2} \end{aligned}$$

# CR3BP

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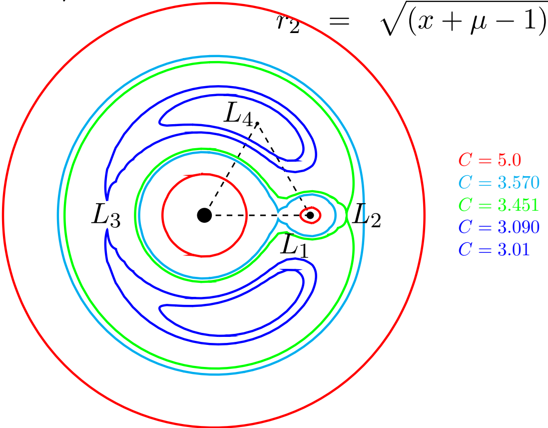
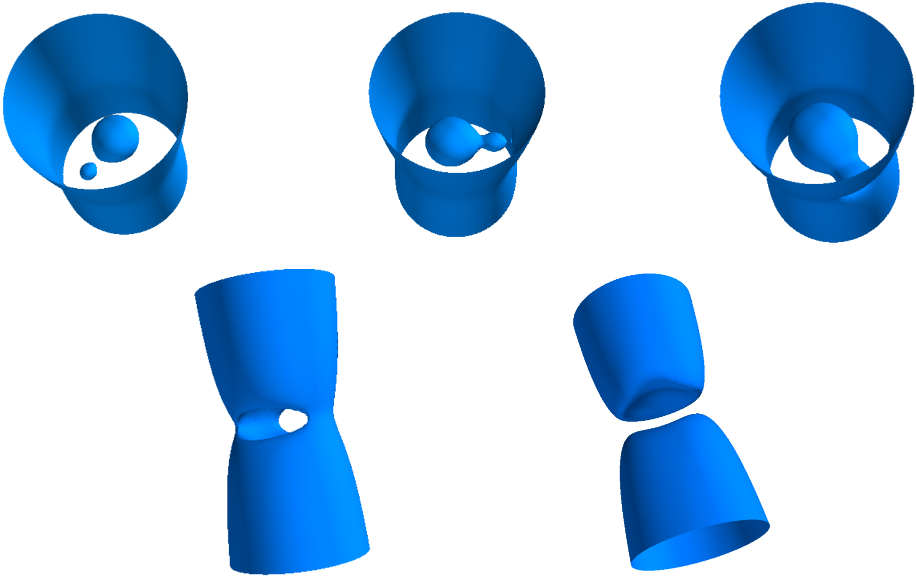


# CR3BP, II

Rotating frame:



# General $N$BP, Scale symmetry

If the functions $\mathbf{r}_i(t)$, $\mathbf{v}_i(t)$, $i = 1, \ldots, N$ is solution, then next solution is also valid

$$\begin{aligned} \boldsymbol{\rho}_i(t) &= \lambda \mathbf{r}_i(\lambda^{-3/2} t) \\ \dot{\boldsymbol{\rho}}_i(t) &= \lambda^{-1/2} \mathbf{v}_i(\lambda^{-3/2} t). \end{aligned}$$

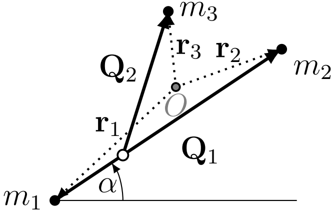
For this solution

$$\begin{aligned} h' &= \lambda^{-1} h \\ J' &= \lambda^{1/2} J, \\ I' &= \lambda^2 I, \end{aligned}$$

This symmetry allow us to limit ourselves to only three values of constant energy, for example, $h = -1/2, 0, 1/2$.

# General 3BP, Jacobi coordinates

In this case the dimension of configuration space can be reduced to 3.
To do this let us factorize the space by transfers and rotations
$\mathbb{R}^6 \to \mathbb{R}^4 : \mathbb{R}^6/\mathbb{T}$.




$$\begin{aligned} \mathbf{Q}_1 &= \mathbf{r}_2 - \mathbf{r}_1, \\ \mathbf{Q}_2 &= \mathbf{r}_3 - \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1+m_2}, \end{aligned}$$

$$T = \frac{1}{2}(m_1\dot{\mathbf{r}}_1^2 + m_2\dot{\mathbf{r}}_2^2 + m_3\dot{\mathbf{r}}_3^2) = \frac{1}{2}\left(\mu_1\dot{\mathbf{Q}}_1^2 + \mu_2\dot{\mathbf{Q}}_2^2\right),$$

$$L = T(\dot{\mathbf{Q}}_1, \dot{\mathbf{Q}}_2) + U(\mathbf{Q}_1, \mathbf{Q}_2),$$

$$J = \mu_1\mathbf{Q}_1 \times \dot{\mathbf{Q}}_1 + \mu_2\mathbf{Q}_2 \times \dot{\mathbf{Q}}_2.$$

Here $\mu_1 = m_1m_2/(m_1+m_2)$, $\mu_2 = m_3(m_1+m_2)/(m_1+m_2+m_3)$.

# $2D$ General 3BP, Hopf transformation

Factorizing by rotations:

$$\begin{aligned}\xi_1 &= \tfrac{1}{2}\mu_1|\mathbf{Q}_1|^2 - \tfrac{1}{2}\mu_2|\mathbf{Q}_2|^2,\\ \xi_2 + i\xi_3 &= \sqrt{\mu_1\mu_2}\,\mathbf{Q}_1\bar{\mathbf{Q}}_2.\end{aligned}$$

$(\xi_1, \xi_2, \xi_3) \in \mathbb{X}$ – 3-dimensional space.

Each element of the space $\mathbb{X}$ is a class of oriented congruent triangles, it is called *form space*. In this space, the length of the vector $(\xi_1, \xi_2, \xi_3)$ is equal to the moment of inertia:

$$\begin{aligned}I &= m_1|\mathbf{r}_1|^2 + m_2|\mathbf{r}_2|^2 + m_3|\mathbf{r}_3|^2 = \frac{m_1m_2r_{12}^2 + m_1m_3r_{13}^2 + m_2m_3r_{23}^2}{M}\\ &= \mu_1|\mathbf{Q}_1|^2 + \mu_2|\mathbf{Q}_2|^2 = \sqrt{\xi_1^2+\xi_2^2+\xi_3^2}.\end{aligned}$$

Kinetic energy $T$ and angular momentum $J$ (conjugate to $\alpha$):

$$\begin{aligned}T &= \frac{4J^2 + \dot{\xi}_1^2 + \dot{\xi}_2^2 + \dot{\xi}_3^2}{8\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}},\\ J &= \mu_1\mathbf{Q}_1 \times \dot{\mathbf{Q}}_1 + \mu_2\mathbf{Q}_2 \times \dot{\mathbf{Q}}_2,\\ &= \sqrt{\xi_1^2+\xi_2^2+\xi_3^2}\frac{d\lambda}{dt} + \frac{\xi_2\dfrac{d\xi_3}{dt} - \xi_3\dfrac{d\xi_2}{dt}}{2(\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}+\xi_1)}\end{aligned}$$

# $2D$ General 3BP, Region of Possible Motion

Hamiltonian

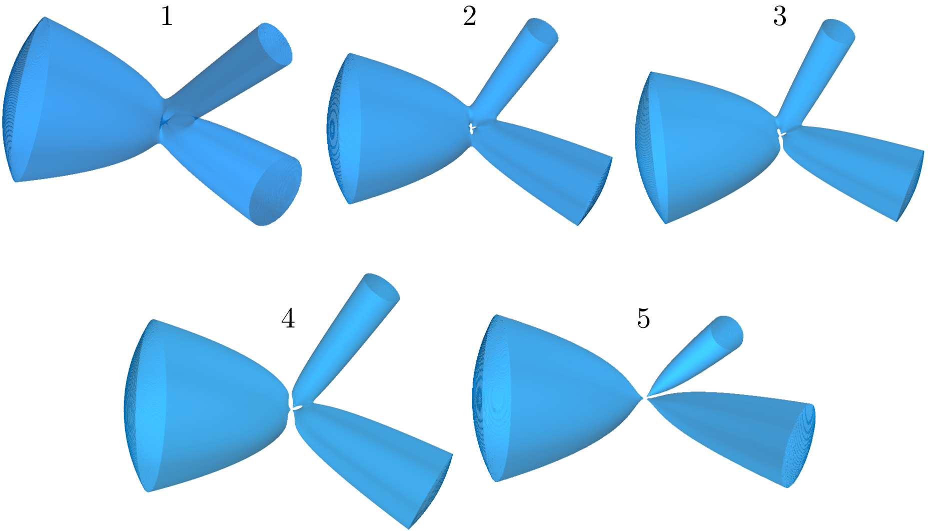
$$H = T - U = \frac{4J^2 + \dot{\xi}_1^2 + \dot{\xi}_2^2 + \dot{\xi}_3^2}{8\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}} - U(\xi_1, \xi_2, \xi_3) = h$$

$$U(\xi_1, \xi_2, \xi_3) = \sum_{i,j=1, i<j}^{3} \frac{m_i m_j}{r_{ij}} = \frac{1}{\sqrt[4]{\xi_1^2 + \xi_2^2 + \xi_3^2}} W(\varphi, \theta),$$

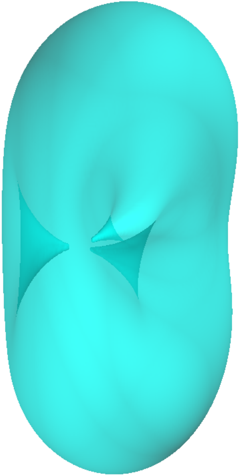
The surface of zero velocity (in $\xi_1 \xi_2 \xi_3$-space):

$$U(\xi_1, \xi_2, \xi_3) + h - \frac{J^2}{2\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}} = 0.$$

# Zero velocity surfaces ($2D$ General Three-Body Problem)



# $2D$ General 3BP, Inner Surface



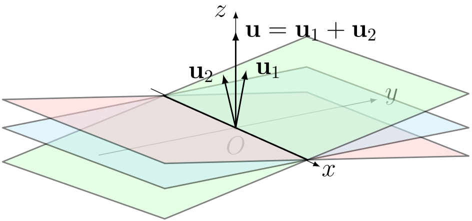
By Sundman's theorem

$$J > 0 \quad \Rightarrow \quad I \geq I_0.$$

# $3D$ General 3BP, What to Do?

We want to reduce $3D$ motion to just considered $2D$ case.

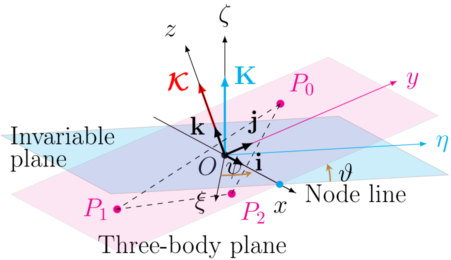
Let us remember the *node elimination*.




Blue plane is invariable Laplacian plane, $\mathbf{u}$ – total angular momentum.
Red and green planes are the planes of osculating orbits of body 1 and body 2 with angular momenta $\mathbf{u_1}$ and $\mathbf{u_2}$ respectively.

$\mathbf{u} = \mathbf{u_1} + \mathbf{u_2}$, therefore the vectors $\mathbf{u}$, $\mathbf{u_1}$ and $\mathbf{u_2}$ are linear dependent, so they lie in one plane; and the red and green planes are intersected with blue plane by the same line orthogonal to all three vectors.

# Levi-Civita reduction

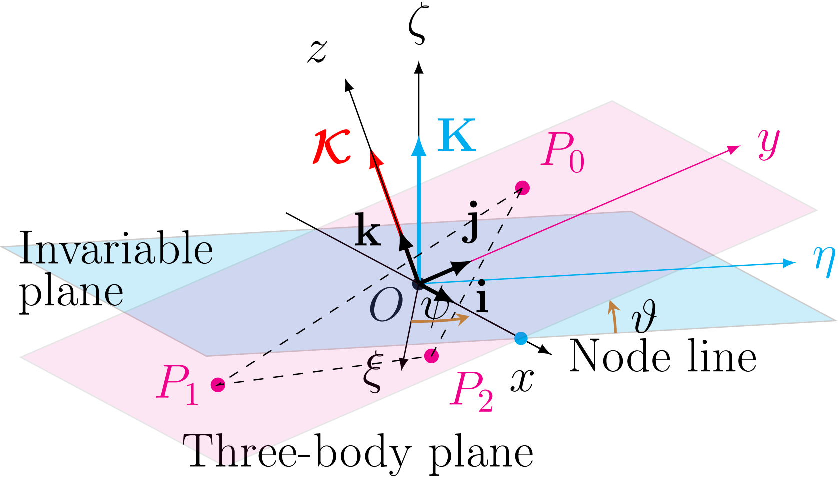



Let us separate the motion of three body plane from motion in this plane. I. The three body plane are defined by angle $\psi$ (angle from abscissa axis in invariable plane to node of three body plane) and angle $\vartheta$ (the inclination of three body plane to invariable plane). Some kind of oblique variables.

II. The coordinates $x_i, y_i$ in the three body plane $Oxy$. Lagrangian coordinates of the system are:

$$\vartheta, \quad \psi,$$
$$x_i, \quad y_i, \qquad i = 0, 1, 2$$

# Levi-Civita reduction



# Levi-Civita reduction, II

$$\begin{aligned} x_i &= x_i^{\mathrm{b}} \cos\psi + y_i^{\mathrm{b}} \sin\psi, \\ y_i &= -(x_i^{\mathrm{b}} \sin\psi - y_i^{\mathrm{b}} \cos\psi) \cos\theta + z_i^{\mathrm{b}} \sin\theta, \\ z_i &= (x_i^{\mathrm{b}} \sin\psi - y_i^{\mathrm{b}} \cos\psi) \sin\theta + z_i^{\mathrm{b}} \cos\theta. \end{aligned}$$

$$\mathbf{v} = (\dot{x}_i - y_i\dot{\psi}\cos\vartheta)\mathbf{i} + (\dot{y}_i + x_i\dot{\psi}\cos\vartheta)\mathbf{j} + (\dot{y}_i\dot{\vartheta} - x_i\dot{\psi}\sin\vartheta)\mathbf{k}$$

$$\begin{aligned} T &= \mathcal{T} + \mathfrak{T} : \quad \mathcal{T} = \frac{1}{2}\sum_{i=1}^{3} m_i \left\{ (\dot{x}_i - y_i\dot{\psi}\cos\vartheta)^2 + (\dot{y}_i + x_i\dot{\psi}\cos\vartheta)^2 \right\}, \\ &\qquad\qquad\quad\;\; \mathfrak{T} = \frac{1}{2}\sum_{i=1}^{3} m_i \left\{ (y_i\dot{\vartheta} - x_i\dot{\psi}\sin\vartheta)^2 \right\}. \end{aligned}$$

# Levi-Civita reduction, III

Conjugate momenta:

$$\begin{aligned} X_i &= \frac{\partial \mathcal{T}}{\partial \dot{x}_i} = m_i(\dot{x}_i - y_i\dot{\psi}\cos\vartheta), \\ Y_i &= \frac{\partial \mathcal{T}}{\partial \dot{y}_i} = m_i(\dot{y}_i + x_i\dot{\psi}\cos\vartheta), \\ \Theta &= \frac{\partial \mathfrak{T}}{\partial \dot{\vartheta}} = \dot{\vartheta}\sum_i m_i y_i^2 + \dot{\psi}\sin\vartheta \sum_i m_i x_i y_i \\ \Psi &= \frac{\partial \mathcal{T}}{\partial \dot{\psi}} + \frac{\partial \mathfrak{T}}{\partial \dot{\psi}}. \end{aligned}$$

$$\Theta = 0, \qquad \Psi = J.$$

$$\begin{aligned} \mathcal{T} &= \frac{1}{2}\sum_i \frac{1}{m_i}(X_i^2 + Y_i^2), \\ \mathfrak{T} &= \frac{1}{2}\sum_i m_i y_i^2 J^2 \sin^2\vartheta / D(x_i, y_i) \\ U &= U(x_i, y_i) \end{aligned}$$

# Jacobi coordinate in the 3-body plane

$$T_q = \frac{\mu_1}{2}(\dot{Q_1}_x^2 + \dot{Q_1}_y^2) + \frac{\mu_2}{2}(\dot{Q_2}_x^2 + \dot{Q_2}_y^2) + \left(\mu_1 {Q_1}_y^2 + \mu_2 {Q_2}_y^2\right)\dot{\theta}^2$$
$$+\left(\left(\mu_1 {Q_1}_x^2 + \mu_2 {Q_2}_x^2\right)\sin^2\theta + \left(\frac{\mu_1}{2}({Q_1}_x^2 + {Q_1}_y^2) + \frac{\mu_2}{2}({Q_2}_x^2 + {Q_2}_y^2)\right)\cos^2\theta\right)\dot{\psi}^2$$
$$-2\dot{\theta}\dot{\psi}\sin\theta\left(\mu_1 {Q_1}_x {Q_1}_y + \mu_2 {Q_2}_x {Q_2}_y\right)$$
$$\dot{\psi}\left(\mu_1\left({Q_1}_x \dot{Q}_{1y} - \dot{Q}_{1x} {Q_1}_y\right) + \mu_2\left({Q_2}_x \dot{Q}_{2y} - \dot{Q}_{2x} {Q_2}_y\right)\right)\cos\theta$$

$$\mathbf{Q}_1, \mathbf{Q}_2 \quad \rightarrow \quad \xi_1, \xi_2, \xi_3$$

# Sundman inequality

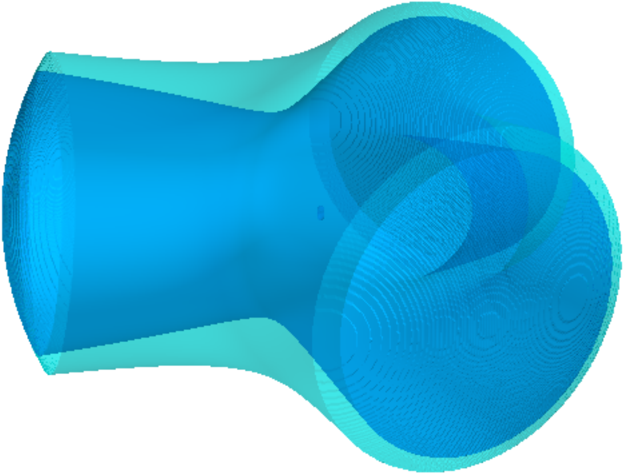
The energy integral of planar three body problem give us the inequality for possible motion (in form space):

$$H = T - U = \frac{4J^2 + \dot{\xi}_1^2 + \dot{\xi}_2^2 + \dot{\xi}_3^2}{8\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}} - U(\xi_1, \xi_2, \xi_3) \leq h$$

This is the same as Sundman enequality:

$$\frac{J^2}{2I} - U(\xi_1, \xi_2, \xi_3) \leq h$$

However, the Sundman inequality holds in three-dimensional space as well, but the inequality is strong in the planar problem only.
For points in possible motion region this inequality is hold.



THANKS

for

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