Levi-Civita reduction in three-body problem

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Abstract. The constructed areas of possible motion of a three-body planar problem in the case of a spatial problem require tools to reduce the problem. To study a spatial problem, the space of forms is reduced to the space of forms of a plane problem. Since the three bodies are always in the same plane, the shape of the triangle is described using a tool already used in the plane problem. Levi-Civita reduction separates the variables responsible for the configuration of the three bodies from the variables describing the motion of the plane of the three bodies.

Introduction

In the planar three-body problem, a zero-velocity surface can be built. By fixing the energy constant ($h < 0$), we can construct such a surface in the form space, that is, in the factor space of the configuration space by transfers and rotation. Depending on the value of the angular momentum J, we get five topologically different types of surface, that is, five topologically different types of the area of possible motion [2].

1. Reduction

One can write the kinetic energy T in a planar problem reduced to the form space ξ_1, ξ_2, ξ_3 as

$$T = \frac{4J^2 + \dot{\xi}_1^2 + \dot{\xi}_2^2 + \dot{\xi}_3^2}{8\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}} \tag{1}$$

and potential function U depends on $U(\xi_1, \xi_2, \xi_3)$ and can be written in the form

$$U = \frac{1}{\rho} D(\theta, \phi), \tag{2}$$

where ρ, θ, ϕ are spherical coordinates in the $\xi_1\xi_2\xi_3$ space.

We want to reduce the problem to the form space.

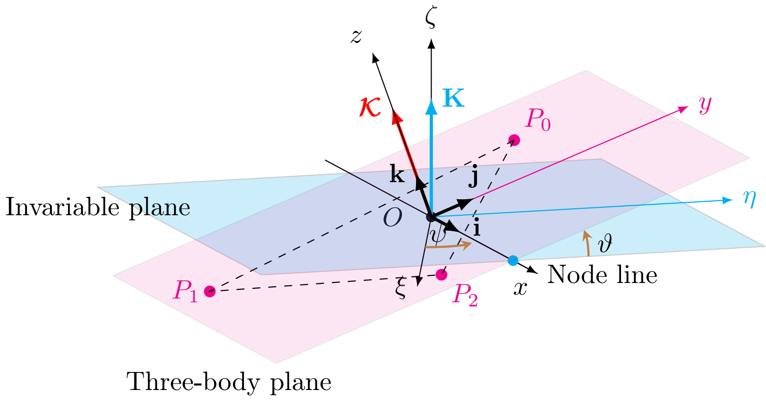

FIGURE 1. Coordinates ψ, ϑ of the plane of three bodies. **K** – vector of total angular momentum

In a planar three-body problem, the kinetic energy is

$$\begin{array}{rcl} T & = & \frac{4J^2+\dot{\xi}_1^2+\dot{\xi}_2^2+\dot{\xi}_3^2}{8\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}}, \\ J & = & \sqrt{\xi_1^2+\xi_2^2+\xi_3^2}\frac{d\lambda}{dt} + \frac{\xi_2\frac{d\xi_3}{dt}-\xi_3\frac{d\xi_2}{dt}}{2(\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}+\xi_1)} \end{array} \tag{3}$$

To extend a planar problem to a spatial one, we will use the Levi-Civita reduction [1]. In Fig. 1, the angles ψ and ϑ determine the position of the plane of the three bodies relative to the Laplace invariable plane. Then the positions of the points are determined by these angles and the coordinates of the points in this plane.

2. Equations of motion

Let us express kinetic energy in the variables ψ, ϑ and x_i, y_i, z_i. For the coordinates x_i, y_i, z_i we have

$$\begin{array}{rcl} x_i & = & X_i\cos\psi + Y_i\sin\psi, \\ y_i & = & (-X_i\sin\psi + Y_i\cos\psi)\cos\vartheta + Z_i\sin\vartheta, \\ z_i & = & (X_i\sin\psi - Y_i\cos\psi)\sin\vartheta + Z_i\cos\vartheta, \end{array} \tag{4}$$

and for kinetic energy T

$$\begin{array}{rcl} T = \mathcal{T} + \mathfrak{T}: \quad \mathcal{T} & = & \frac{1}{2}\sum_{i=1}^{3} m_i\left\{(\dot{x}_i - y_i\dot{\psi}\cos\vartheta)^2 + (\dot{y}_i + x_i\dot{\psi}\cos\vartheta)^2\right\}, \\ \mathfrak{T} & = & \frac{1}{2}\sum_{i=1}^{3} m_i\left\{(y_i\dot{\vartheta} - x_i\dot{\psi}\sin\vartheta)^2\right\}. \end{array} \tag{5}$$

Note that in conjugate momenta $P_i = \frac{\partial T}{\partial \dot{x}_i}$, $Q_i = \frac{\partial T}{\partial \dot{y}_i}$ the expression for $\mathcal{T}$ is equal to the following

$$\mathcal{T} = \frac{1}{2}\sum_{i=0}^{2}(P_i^2 + Q_i^2)/m_i. \tag{6}$$

Conclusion

The coordinates x_i, y_i, z_i define the position in the plane of three bodies, and therefore we can construct a form space similar to the planar three-body problem [2].

References

[1] Tulio Levi-Civita, *Sulla riduzione del problema dei tre corpi.* Atti Ist. Veneto di Sc., lett. ed arti, t. LXXIV, 1915.

[2] V. B. Titov, *Zero-Velocity Surface in the General Three-Body-Problem.* Vestnik St. Petersburg University, Mathematics, 2023, Vol. 56, No. 1.

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