Reversible difference schemes for classical nonlinear oscillators

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Abstract. The report considers the properties of finite-difference schemes for classical oscillators that define a one-to-one correspondence between initial and final values (reversible difference schemes). The results of computer experiments with these schemes and their justification are given.

At PCA'14 [1], we started with a puzzle: why are finitely integrable dynamical systems integrable in elliptic functions? Strictly speaking, the answer to this question was not found in the continuous theory, but Painlevé made an important observation: any dynamical system that defines a birational correspondence between initial and final values on algebraic integral varieties is integrable in classical transcendental functions, usually in elliptic ones.

When we proceed to finite differences, the situation changes radically. In our talk at PCA'21 [2], it was shown that any dynamical system with a quadratic right-hand side admits a reversible difference scheme, i.e., a scheme that defines a birational correspondence between the initial and final positions of the system. In this case, there is no need to restrict the consideration to the integral manifold, since difference schemes define the Cremona transformations. Approximate solution of the Cauchy problem

$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}), \quad \vec{x}(0) = \vec{x}_0 \tag{1}$$

is obtained by successively applying the Cremona transformation $\vec{R}$ to point $\vec{x}_0$:

$$\vec{x}_n = \vec{R}^n \vec{x}_0 \simeq \vec{x}(n\Delta t).$$

Reversible difference schemes can be constructed for a wide class of nonlinear dynamic with quadratic right-hand side, which includes both all classical nonlinear oscillators integrable in elliptic functions and systems that are not integrable in classical transcendental functions, e.g., asymmetric tops.

In the computer experiments presented in [2], we were surprised to see that the points of approximate solutions found by reversible schemes for classical oscillators line up into curves. This report will present the results of the study of these curves.

Elliptic oscillators correspond to the special case, when the points of not only exact but also approximate solutions lie on elliptic curves. We have written out the equations of these curves explicitly for the Jacobi oscillator in [3], having obtained a kind of difference analogue of the Lagutinski theory.

The curves themselves depend on the time sampling step Δt. The reduction of the Cremona transformation to integral curves (which are inevitably invariants of this transformation) defines a birational transformation on an elliptic curve. This transformation is always described using elliptic integrals of the first type, which gives us a description of the approximate solution in the form of a quadrature

$$\int\limits_{\vec{x}_n}^{\vec{x}_{n+1}} v(\vec{x}, \Delta t) dx_1 = \Delta t,$$

where vdx_1 is an elliptic integral of the first kind on the appropriate elliptic curve. This description is quite analogous to that obtained in the continuous theory by separation of variables.

The appearance of quadrature leads to the periodicity of motion. For approximate solutions, the very concept of periodicity can be introduced in different ways. First of all, we have shown how to choose the step Δt so that the solution is a periodic sequence of points with a given period. We then showed that the approximate solution can be represented as a set of values of a meromorphic doubly periodic function:

$$\vec{x}_n = \wp(n\Delta t).$$

Thus, the appearance of integral curves, on which the points of approximate solutions lie, leads to the periodicity of the approximate solution.

The discrete and continuous theories of elliptic oscillators are described by the same formulas: the quadrature describes the transition from initial to final data, the motion is periodic, it is described by meromorphic functions, and so on. The whole difference lies in the fact that in the discrete theory the birational transformation describing the transition from the old position of the system to the new one is continued to the Cremona transformation. Hermite was absolutely right in assuming that Cremona's theory of transformations encompasses the theory of elliptic functions. This is a special case of the general theory of dynamical systems approximated by reversible difference schemes, i.e. Cremona transformations.

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