Asymptotic forms of solutions to system of nonlinear partial differential equations

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Abstract. In [1, 2] we considerably develop the methods of power geometry for a system of partial differential equations and apply them to computing the asymptotic forms of solutions to the problem of evolution of the turbulent flow. For each equation of the system, its Newton polyhedron and its hyperfaces with their normals and truncated equations are calculated. To simplify the truncated systems, power-logarithmic transformations are used and the truncated systems are further extracted. Results: (1) the boundary layer on the needle is absent in liquid, while in gas it is described in the first approximation; (2) one-dimensional model of evolution of turbulent bursts have eight asymptotic forms, presented explicitly.

1. Introduction

A universal asymptotic nonlinear analysis is formed, whose unified methods allow finding asymptotic forms and expansions of solutions to nonlinear equations and systems of different types: Algebraic; Ordinary differential equations (ODEs); Partial differential equations (PDEs).

This calculus contains two main methods: 1) *Transformation of coordinates, bringing equations to normal form*; 2) *Separating truncated equations.*

Two kinds of coordinate changes can be used to analyze the resulting equations: A) *Power*; B) *Logarithmic.*

Here, we consider systems of nonlinear partial differential equations in two variants:

a) *with solvable truncated system*; b) *without solvable truncated system.* We show how to find asymptotic forms of their solutions using algorithms of power geometry. In this case, by asymptotic form of solution, we mean a simple expression in which each of the independent or dependent variables tends to zero or infinity.

Here, we consider two fluids problems: (a) boundary layer and (b) turbulence flow by methods of power geometry.

For problem (a), it was firstly given in [3, Chapter 6, Section 6]; see also [4, 5]. A boundary layer on a needle has a stronger singularity than on a plane, and it was first considered in [5].

For problem (b), we firstly make it in [1, 2] and we are not sure that it can be solved with the usual analysis.

The structure of the paper is as follows. Section 2 outlines the basics of power geometry for partial differential equations. In Section 3, the theory and algorithms are further developed to apply to variant (b) problems. In Section 4, they are used to compute asymptotic forms of evolution of turbulent flow.

2. Basics of Power Geometry [3, Chapters VI–VIII]

Let $X = (x_1, \ldots, x_m) \in \mathbb{C}^m$ be independent and $Y = (y_1, \ldots, y_n) \in \mathbb{C}^n$ be dependent variables. Place $Z = (X, Y) \in \mathbb{C}^{n+m}$. *Differential monomial* $a(Z)$ is a product of an ordinary monomial $cZ^R = cz_1^{r_1} \cdots z_{m+n}^{r_{m+n}}$, where $c = \text{const} \in \mathbb{C}$, and a finite number of derivatives of the form

$$\frac{\partial^l y_j}{\partial x_1^{l_1} \cdots \partial^{l_m} x_m} \equiv \frac{\partial^l y_j}{\partial X^L}, \quad l_j \geq 0, \sum_{j=1}^{m} l_j = l, \quad L = (l_1, \ldots, l_m)\,. \tag{1}$$

The differential monomial $a(Z)$ corresponds to its vector exponent of degree $Q(a) \in \mathbb{R}^{m+n}$, formed by the following rules:

$$Q\left(Z^R\right) = R, \quad Q\left(\partial^l y_j / \partial X^L\right) = (-L, E_j), \tag{2}$$

where E_j is the unit vector. The product of monomials corresponds to the sum of their vector exponents of degree: $Q(ab) = Q(a) + Q(b)$. *Differential sum* is the sum of differential monomials:

$$f(Z) = \sum a_k(Z). \tag{3}$$

The set $\mathbf{S}(f)$ of vector exponents $Q(a_k)$ is called *support of sum* $f(Z)$. The closure of the convex hull

$$\mathbf{\Gamma}(f) = \left\{ Q = \sum \lambda_j Q_j, Q_j \in \mathbf{S}, \lambda_j \geq 0, \sum \lambda_j = 1 \right\}$$

of the support $\mathbf{S}(f)$ is called the *polyhedron of the sum* $f(Z)$. The boundary $\partial\mathbf{\Gamma}$ of the polyhedron $\mathbf{\Gamma}(f)$ consists of generalized faces $\mathbf{\Gamma}_j^{(d)}$, where $d = \dim \mathbf{\Gamma}_j^{(d)}$, $0 \leq d \leq m + n - 1$. Each face $\mathbf{\Gamma}_j^{(d)}$ corresponds to:

- *Normal cone:* $\mathbf{U}_j^{(d)} = \{P \in \mathbb{R}_*^{m+n} : \langle P, Q' \rangle = \langle P, Q'' \rangle > \langle P, Q''' \rangle\,, \}$, where $Q', Q'' \in \mathbf{\Gamma}_j^{(d)}, Q''' \in \mathbf{\Gamma} \backslash \mathbf{\Gamma}_j^{(d)}$, and the space $\mathbb{R}_*^{m+n}$ is conjugate to the space $\mathbb{R}^{m+n}$ and $\langle \cdot, \cdot \rangle$ is a scalar product;
- *Truncated sum:* $\hat{f}_j^{(d)}(Z) = \sum a_k(Z)$ over $Q(a_k) \in \mathbf{\Gamma}_j^{(d)} \bigcap \mathbf{S}$.

Consider a system of equations:

$$f_i(X, Y) = 0, \quad i = 1, \ldots, n, \tag{4}$$

where f_i are differential sums. Each equation $f_i = 0$ corresponds to: its support $\mathbf{S}(f_i)$; its polyhedron $\mathbf{\Gamma}(f_i)$ with a set of faces $\mathbf{\Gamma}_{ij}^{(d_i)}$ in the main space $\mathbb{R}^{m+n}$; set of their normal cones $\mathbf{U}_{ij}^{(d_i)}$ in the dual space $\mathbb{R}_*^{m+n}$; set of truncated equations $\hat{f}_{ij}^{(d_i)}(X,Y) = 0$.

The set of truncated equations

$$\hat{f}_{ij_i}^{(d_i)}(X,Y) = 0, \quad i = 1,\dots,n, \tag{5}$$

is a *truncated system* if the intersection

$$\mathbf{U}_{1j_i}^{(d_1)} \cap \dots \cap \mathbf{U}_{nj_n}^{(d_n)}. \tag{6}$$

is not empty. A truncated system is always a quasi-homogeneous system.

In the solution of the system (4),

$$y_i = \varphi_i(X), \quad i = 1,\dots,n, \tag{7}$$

where φ_i are series in powers of x_k and their logarithms, each φ_i corresponds to its support, polyhedron, normal cones $\mathbf{u}_i$, and truncations. Here, the logarithm $\ln x_i$ has a zero exponent of degree on x_i. The set of truncated solutions $y_i = \hat{\varphi}_i$, $i = 1,\dots,n$, corresponds to the intersection of their normal cones: $\mathbf{u} = \bigcap_{i=1}^n \mathbf{u}_i \subset \mathbb{R}_*^{m+n}$. If it is not empty, it corresponds to *truncated solution*: $y_i = \hat{\varphi}_i$, $i = 1,\dots,n$.

Theorem 1. *If the normal cone* $\mathbf{u}$ *intersects the normal cone* (6)*, then the truncation* $y_i = \hat{\varphi}_i(X)$*,* $i = 1,\dots,n$*, of this solution satisfies the truncated system* (5).

Multiplying the differential sum (5) with the support $\mathbf{S}(f)$ by the monomial Z^R gives the differential sum, $g(Z) = Z^R f(Z)$, with the support $\mathbf{S}(g) = R + \mathbf{S}(f)$. Thus, the multiplication leads to a shift of supports. Multiplications by monomials form a group of linear transformations of supports, and they can be used to simplify supports, differential sums, and systems of equations.

3. Algorithms of power geometry and their implementation

A matrix α is called *unimodular* if all its elements are integer and $\det \alpha = \pm 1$.

Problem 1. *Let n-dimensional integer vector $A = (a_1, a_2, \dots, a_n)$ be given. Find an n-dimensional unimodular matrix α such that the vector $A\alpha = C = (c_1,\dots,c_n)$ contains only one coordinate c_n different from zero.*

Its solution was given in [6, 7, 8].

Transformation of the variables

$$\ln W = (\ln Z)\,\alpha, \ \text{where } \alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ 0 & \alpha_{22} \end{pmatrix}, \tag{8}$$

is called *power transformation*, where α_{11}, α_{22} are square matrices of sizes m and n, respectively and $\ln Z = (\ln z_1,\dots,\ln z_{m+n})$.

Theorem 2 ([3]). *The power transformation* (8) *changes a differential monomial $a(Z)$ with exponent of degree $Q(a)$ into a differential sum $b(W)$ with exponent of degree $Q(b)$:*

$$R = Q(b) = Q(a)\alpha^{-1*}, \tag{9}$$

where $*$ denotes transposition.

Theorem 3 ([3]). *If the system* (4) *is a quasi-homogeneous system and $d = \dim \tilde{\mathbf{\Gamma}}$, then there exist a power transformation* (8) *and monomials Z^{T_i}, $i = 1, \ldots, n$ which change the system* (4) *into the system $g_i(W) \equiv Z^{T_i} f_i(Z) = 0$, $i = 1, \ldots, n$, where all $g_i(W)$ are differential sums, and all their supports $\mathbf{S}(g_i)$ have $m+n-d$ identical coordinates q_j equal to zero.*

Transformation

$$\zeta_j = \ln z_j \tag{10}$$

is called *logarithmic transformation.*

Theorem 4 ([9]). *Let $f(Z)$ be such a differential sum that for all its monomials, jth component of q_j vector degree exponent $Q = (q_1, \ldots, q_{m+n})$ is zero, then as a result of the logarithmic transformation* (10), *a differential sum $f(Z)$ transforms into a differential sum from $z_1, \ldots, \zeta_j, \ldots, z_n$.*

For $z_j \to 0$ or ∞, the coordinate $\zeta_j = \ln w_j$ always tends to $\pm\infty$. If we are interested only in those solutions (7) which have a normal cone $\mathbf{u}$ intersecting a given cone K, then the cone K is called the *cone of problem.* Thus, after the logarithmic transformation (10) for the coordinate ζ_j in the cone of the problem, we have $p_j \geq 0$.

In the following, we will not consider all possible truncated systems (5), but only those in which one of the equations has dimension $d_i = m + n - 1$. The calculations show that in this case the above procedure will cover all the truncated systems. Finally, it is convenient to combine the power and logarithmic transformations.

The CAS `Maple 2021` was used for calculations in this work. A library of procedures based on the `PolyhedralSets` CAS `Maple` package was developed to implement the algorithms of power geometry. The library includes calculation procedures:

- vector power exponent Q of the differential monomial $a(Z)$ for a given order of independent and dependent variables;
- support $\mathbf{S}$ of a partial differential equation written as a sum of differential monomials;
- Newton's polyhedron $\mathbf{\Gamma}$ in the form of a graph of generalized faces $\mathbf{\Gamma}_j^{(d)}$ of all dimensions d for the given support of the equation ; the number j is given by the program; each generalized face has its own number j; each line of the graph contains all generalized faces $\mathbf{\Gamma}_j^{(d)}$ of the same dimension d, the first line contains the Newton's polyhedron $\mathbf{\Gamma}$, the next line contains all faces $\mathbf{\Gamma}_j^{(m+n-1)}$ of dimension $m+n-1$ and so on; the last line contains the empty

set; if $\mathbf{\Gamma}_j^{(d)} \subset \mathbf{\Gamma}_k^{(d+1)}$, then they are connected by an arrow; in [3, Ch. 1, Section 1], "the structural diagram" was used that is similar to the graph and differs from it in two properties: numeration of faces $\mathbf{\Gamma}_j^{(d)}$ is independent for each dimension d and arrows are replaced by segments (see also [10]);

- normal vector N_j for the each generalized face $\mathbf{\Gamma}_j^{(m+n-1)}$ for the second line of the graph;
- truncated equation $\hat{f}_j^{(d)} = 0$ by the given number j of the generalized face or by a given normal vector N_j;
- normal cone of the corresponding generalized face: if the face

$$\mathbf{\Gamma}_j^{(d)} = \mathbf{\Gamma}_i^{(m+n-1)} \cap \mathbf{\Gamma}_k^{(m+n-1)} \cap \cdots \cap \mathbf{\Gamma}_l^{(m+n-1)},$$

  then the normal cone $\mathbf{U}_j^{(d)}$ is the conic hull of the normals $N_i, N_k, \ldots, N_l$;
- power or logarithmic transformation of the original variables by a given normal N of the hyperface. For this purpose, the algorithms for constructing the unimodular matrix described in [6, 7, 8] are used.

4. The k–ε Model of Evolution of Turbulent Bursts

According to [11, 12, 13], the model is described by the system

$$\begin{aligned} k_t &= \left(\frac{k^2}{\varepsilon} k_x\right)_x - \varepsilon, \\ \varepsilon_t &= \left(\frac{k^2}{\varepsilon} \varepsilon_x\right)_x - \gamma \frac{\varepsilon^2}{k}. \end{aligned} \tag{11}$$

Here, time t and coordinate x are independent variables, the turbulent density k and the dissipation rate ε are dependent variables, and γ is a real parameter. Here, $m = n = 2$, $m + n = 4$ and $x_1 = t$, $x_2 = x$, $y_1 = k$, $y_2 = \varepsilon$.

In [1, 2] equations (11) are written as differential sums, such truncated systems are selected, which have one 3-dimensional equation, power and logarithmic transformations are applied and more simple systems are obtained. If they are not solvable, the computations are repeated till solvable systems are obtained, Their solutions, written in initial coordinates, are asymptotic forms of solutions to initial system.

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