# Asymptotic forms of solutions to system of nonlinear partial differential equations

Alexander Batkhin[1,2], Alexander D. Bruno[1]
batkhin@gmail.ru, abruno@keldysh.ru

[1] *Keldysh Institute of Applied Mathematics of RAS*
[2] *Moscow Institute of Physics and Technology*

PCA-2023, April 17–22, 2023

## Talk outlook

1. Introduction
2. Basics of power geometry
3. Algorithms of power geometry and their implementations
4. The $k$–$\varepsilon$ model of evolution of turbulent bursts
5. Asymptotic forms of solutions to the system $S(2)$
6. Summary of results for the $k$–$\varepsilon$ model

## Abstract

In [Bruno, Batkhin, 2022; 2023] we considerably develop the methods of power geometry for a system of partial differential equations and apply them to computing the asymptotic forms of solutions to the problem of evolution of the turbulent flow. For each equation of the system, its Newton polyhedron and its hyperfaces with their normals and truncated equations are calculated. To simplify the truncated systems, power-logarithmic transformations are used and the truncated systems are further extracted. Results:

1. the boundary layer on the needle is absent in liquid, while in gas it is described in the first approximation;
2. one-dimensional model of evolution of turbulent bursts have eight asymptotic forms, presented explicitly.

1. Introduction

2. Basics of power geometry

3. Algorithms of power geometry and their implementations

4. The $k$–$\varepsilon$ model of evolution of turbulent bursts

5. Asymptotic forms of solutions to the system $S(2)$

6. Summary of results for the $k$–$\varepsilon$ model

## Introduction (1)

A universal asymptotic nonlinear analysis is formed, whose unified methods allow finding asymptotic forms and asymptotic expansions of solutions to nonlinear equations and systems of different types:

- *Algebraic*
- *Ordinary differential equations* (ODEs)
- *Partial differential equations* (PDEs)

This calculus contains two main methods:

1. *Transformation of coordinates*, bringing equations to *normal form*
2. *Separating truncated equations*

## Introduction (2)

Two kinds of coordinate changes can be used to analyze the resulting equations:

A) *Power*
B) *Logarithmic*

Here, we consider systems of nonlinear partial differential equations in two variants:

a) with *solvable truncated system*
b) without *solvable truncated system*

We show how to find asymptotic forms of their solutions using algorithms of power geometry. In this case, by asymptotic form of solution, we mean a simple expression in which each of the independent or dependent variables tends to zero or infinity.

## Introduction (3)

Here, we consider two fluids problems:

(a) boundary layer and

(b) turbulence flow by methods of power geometry.

For problem **(a)**, it was firstly given in [Bruno, 2000, Chapter 6, Section 6]; see also [Blasius, 1908; Bruno, Shadrina, 2007]. A boundary layer on a needle has a stronger singularity than on a plane, and it was first considered in [Bruno, Shadrina, 2007].

For problem **(b)**, we firstly make it in [Bruno, Batkhin, 2022; 2023] and we are not sure that it can be solved with the usual analysis.

## The structure of the talk

- Section 2 outlines the *basics of power geometry* for partial differential equations

## The structure of the talk

- Section 2 outlines the ***basics of power geometry*** for partial differential equations
- In Section 3, the theory and algorithms are ***further developed*** to apply to variant (b) problems.

## The structure of the talk

- Section 2 outlines the ***basics of power geometry*** for partial differential equations
- In Section 3, the theory and algorithms are ***further developed*** to apply to variant (b) problems.
- In Section 4 we consider the $k$-$\varepsilon$ ***model of evolution of turbulent bursts*** and simplify it by power and logarithmic transformations.

## The structure of the talk

- Section 2 outlines the ***basics of power geometry*** for partial differential equations
- In Section 3, the theory and algorithms are ***further developed*** to apply to variant (b) problems.
- In Section 4 we consider the $k$-$\varepsilon$ ***model of evolution of turbulent bursts*** and simplify it by power and logarithmic transformations.
- In Section 5 we show a way of ***further simplification*** of equations using the selections of truncated systems.

## The structure of the talk

- Section 2 outlines the ***basics of power geometry*** for partial differential equations
- In Section 3, the theory and algorithms are ***further developed*** to apply to variant (b) problems.
- In Section 4 we consider the $k$-$\varepsilon$ ***model of evolution of turbulent bursts*** and simplify it by power and logarithmic transformations.
- In Section 5 we show a way of ***further simplification*** of equations using the selections of truncated systems.
- In Section 6 we give a ***list of all asymptotic forms*** in the initial coordinates.

This talk is based on

- *Bruno A. D., Batkhin A. B.* Computation of asymptotic forms of solutions to system of nonlinear partial differential equations. // Preprints of KIAM. 2022. No. 48. P. 36. (in Russian) and
- *Bruno A. D., Batkhin A. B.* Asymptotic forms of solutions to system of nonlinear partial differential equations. // Universe. 2023. Vol. 9, no. 1. P. 35 (open access)

1. Introduction

2. Basics of power geometry

3. Algorithms of power geometry and their implementations

4. The $k$–$\varepsilon$ model of evolution of turbulent bursts

5. Asymptotic forms of solutions to the system $S(2)$

6. Summary of results for the $k$–$\varepsilon$ model

# Basics of power geometry (1)

For more detail, see [Bruno, 2000, Chapters VI–VIII]

Let $X = (x_1, \dots, x_m) \in \mathbb{C}^m$ be ***independent*** and $Y = (y_1, \dots, y_n) \in \mathbb{C}^n$ be ***dependent*** variables. Place $Z = (X, Y) \in \mathbb{C}^{n+m}$

***Differential monomial*** $a(Z)$ is a product of an ordinary monomial $cZ^R = cz_1^{r_1} \cdots z_{m+n}^{r_{m+n}}$, where $c = \text{const} \in \mathbb{C}$, and a finite number of derivatives of the form

$$\frac{\partial^l y_j}{\partial x_1^{l_1} \cdots \partial^{l_m} x_m} \equiv \frac{\partial^l y_j}{\partial X^L}, \quad l_j \geqslant 0, \sum_{j=1}^{m} l_j = l, \quad L = (l_1, \dots, l_m)\,.$$

## Basics of power geometry (2)

The differential monomial $a(Z)$ corresponds to its vector exponent of degree $Q(a) \in \mathbb{R}^{m+n}$, formed by the following rules:

$$Q\left(Z^R\right) = R, \quad Q\left(\partial^l y_j / \partial X^L\right) = (-L, E_j),$$

where $E_j$ is the unit vector.

The product of monomials corresponds to the sum of their vector exponents of degree:

$$Q(ab) = Q(a) + Q(b)$$

## Basics of power geometry (3)

***Differential sum*** is the sum of differential monomials:

$$f(Z) = \sum a_k(Z).$$

The set $\mathbf{S}(f) \subset \mathbb{R}^{m+n}$ of vector exponents of degrees $Q(a_k)$ is called ***support of sum*** $f(Z)$. The closure of the convex hull

$$\mathbf{\Gamma}(f) = \left\{ Q = \sum \lambda_j Q_j, Q_j \in \mathbf{S}, \lambda_j \geqslant 0, \sum \lambda_j = 1 \right\}$$

of the support $\mathbf{S}(f)$ is called the ***polyhedron of the sum*** $f(Z)$.

## Basics of power geometry (4)

The boundary $\partial\mathbf{\Gamma}$ of the polyhedron $\mathbf{\Gamma}(f)$ consists of generalized faces $\mathbf{\Gamma}_j^{(d)}$, where $d = \dim \mathbf{\Gamma}_j^{(d)}$, $0 \leqslant d \leqslant m+n-1$. Each face $\mathbf{\Gamma}_j^{(d)}$ corresponds to:

- ***Normal cone:***

$$\mathbf{U}_j^{(d)} = \left\{P \in \mathbb{R}_*^{m+n} : \left\langle P, Q' \right\rangle = \left\langle P, Q'' \right\rangle > \left\langle P, Q''' \right\rangle \right\},$$

  where $Q', Q'' \in \mathbf{\Gamma}_j^{(d)}, Q''' \in \mathbf{\Gamma} \backslash \mathbf{\Gamma}_j^{(d)}$, and the space $\mathbb{R}_*^{m+n}$ is conjugate to the space $\mathbb{R}^{m+n}$ and $\langle \cdot, \cdot \rangle$ is a scalar product;

- ***Truncated sum:***

$$\hat{f}_j^{(d)}(Z) = \sum a_k(Z) \text{ over } Q(a_k) \in \mathbf{\Gamma}_j^{(d)} \bigcap \mathbf{S}.$$

## Basics of power geometry (5)

Consider a system of equations:

$$f_i(X,Y) = 0, \quad i = 1, \dots, n, \tag{1}$$

where $f_i$ are differential sums. Each equation $f_i = 0$ corresponds to:

- its support $\mathbf{S}(f_i)$;
- its polyhedron $\mathbf{\Gamma}(f_i)$ with a set of faces $\mathbf{\Gamma}_{ij}^{(d_i)}$ in the main space $\mathbb{R}^{m+n}$;
- set of its normal cones $\mathbf{U}_{ij}^{(d_i)}$ in the dual space $\mathbb{R}_*^{m+n}$;
- set of truncated equations $\hat{f}_{ij}^{(d_i)}(X,Y) = 0$.

## Basics of power geometry (6)

The set of truncated equations

$$\hat{f}_{ij_i}^{(d_i)}(X,Y) = 0, \quad i = 1, \dots, n, \tag{2}$$

is a ***truncated system*** if the intersection

$$\mathbf{U}_{1j_i}^{(d_1)} \cap \dots \cap \mathbf{U}_{nj_n}^{(d_n)}$$

is not empty. A truncated system is always a ***quasi-homogeneous*** system.

## Basics of power geometry (7)

In the solution of the system (1),

$$y_i = \varphi_i(X), \quad i = 1, \ldots, n, \tag{3}$$

where $\varphi_i$ are series in powers of $x_k$ and their logarithms, each $\varphi_i$ corresponds to its support, polyhedron, normal cones $\mathbf{u}_i$, and truncations. Here, the logarithm $\ln x_i$ has a zero exponent of degree on $x_i$.

## Basics of power geometry (8)

The set of truncated solutions $y_i = \hat{\varphi}_i$, $i = 1, \ldots, n$, corresponds to the intersection of their normal cones:

$$\mathbf{u} = \bigcap_{i=1}^{n} \mathbf{u}_i \subset \mathbb{R}_*^{m+n}.$$

If it is not empty, it corresponds to ***truncated solution***:

$$y_i = \hat{\varphi}_i, \quad i = 1, \ldots, n.$$

### Theorem 1.

*If the normal cone* $\mathbf{u}$ *intersects the normal cone* $\mathbf{U}_{1j_i}^{(d_1)} \cap \cdots \cap \mathbf{U}_{nj_n}^{(d_n)}$*, then the truncation* $y_i = \hat{\varphi}_i(X)$*,* $i = 1, \ldots, n$*, of this solution satisfies the truncated system* $\hat{f}_{ij_i}^{(d_i)}(X, Y) = 0$*,* $i = 1, \ldots, n$*.*

## Basics of power geometry (9)

Multiplying the differential sum (2) with the support $\mathbf{S}(f)$ by the monomial $Z^R$ gives the differential sum, $g(Z) = Z^R f(Z)$, with the support $\mathbf{S}(g) = R + \mathbf{S}(f)$. Thus, the multiplication leads to a shift of supports. Multiplications by monomials form a group of linear transformations of supports, and they can be used to simplify supports, differential sums, and systems of equations

1. Introduction

2. Basics of power geometry

3. Algorithms of power geometry and their implementations

4. The $k$–$\varepsilon$ model of evolution of turbulent bursts

5. Asymptotic forms of solutions to the system $S(2)$

6. Summary of results for the $k$–$\varepsilon$ model

# Algorithms of power geometry (1)

A matrix $\alpha$ is called ***unimodular*** if all its elements are integer and $\det \alpha = \pm 1$.

## Problem 1.

Let $n$-dimensional integer vector $A = (a_1, a_2, \ldots, a_n)$ be given. Find an $n$-dimensional unimodular matrix $\alpha$ such that the vector $A\alpha = C = (c_1, \ldots, c_n)$ contains only one coordinate $c_n$ different from zero.

Its solution was given in [Bruno, Azimov, 2022; 2023a,b].

## Algorithms of power geometry (2)

Transformation of the variables

$$\ln W = (\ln Z)\,\alpha, \text{ where } \alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ 0 & \alpha_{22} \end{pmatrix}, \tag{4}$$

is called ***power transformation***, where $\alpha_{11}$, $\alpha_{22}$ are square matrices of sizes $m$ and $n$, respectively and $\ln Z = (\ln z_1, \dots, \ln z_{m+n})$.

## Algorithms of power geometry (3)

**Theorem 2 ([Bruno, 2000]).**

*The power transformation* (4) *changes a differential monomial $a(Z)$ with exponent of degree $Q(a)$ into a differential sum $b(W)$ with exponent of degree $Q(b)$:*

$$R = Q(b) = Q(a)\alpha^{-1*},$$

*where $*$ denotes transposition.*

## Algorithms of power geometry (4)

**Theorem 3 ([Bruno, 2000]).**

*If the system* (1) *is a quasi-homogeneous system and* $d = \dim \tilde{\mathbf{\Gamma}}$, *then there exist a power transformation* (4) *and monomials* $Z^{T_i}$, $i = 1, \dots, n$ *which change the system* (1) *into the system*

$$g_i(W) \equiv Z^{T_i} f_i(Z) = 0, \quad i = 1, \dots, n,$$

*where all* $g_i(W)$ *are differential sums, and all their supports* $\mathbf{S}(g_i)$ *have* $m + n - d$ *identical coordinates* $q_j$ *equal to zero.*

Transformation

$$\zeta_j = \ln z_j \tag{5}$$

is called ***logarithmic transformation***.

## Algorithms of power geometry (5)

**Theorem 4 ([Bruno, 1996]).**

*Let $f(Z)$ be such a differential sum that for all its monomials, $j$-th component $q_j$ of vector degree exponent $Q = (q_1, \ldots, q_{m+n})$ is zero, then as a result of the logarithmic transformation* (5)*, a differential sum $f(Z)$ transforms into a differential sum from $z_1, \ldots, \zeta_j, \ldots, z_n$.*

For $z_j \to 0$ or $\infty$, the coordinate $\zeta_j = \ln w_j$ always tends to $\pm\infty$. If we are interested only in those solutions (3) which have a normal cone $\mathbf{u}$ intersecting a given cone $K$, then the cone $K$ is called the ***cone of problem***. Thus, after the logarithmic transformation (5) for the coordinate $\zeta_j$ in the cone of the problem, we have $p_j \geq 0$.

## Algorithms of power geometry (6)

In the following, we will not consider all possible truncated systems (2), but only those in which one of the equations has dimension $d_i = m + n - 1$, i.e. the dimension of its Newton polyhedron. The calculations show that in this case the above procedure will cover all the truncated systems. Finally, it is convenient to combine the power and logarithmic transformations.

## Implementation of power geometry algorithms (1)

The CAS `Maple 2021` was used for calculations in this work. A library of procedures based on the `PolyhedralSets` CAS `Maple` package was developed to implement the algorithms of power geometry. The library includes calculation procedures:

- vector power exponent $Q$ of the differential monomial $a(Z)$ for a given order of independent and dependent variables;
- support $\mathbf{S}$ of a partial differential equation written as a sum of differential monomials;

## Implementation of power geometry algorithms (2)

- Newton's polyhedron $\mathbf{\Gamma}$ in the form of a graph of generalized faces $\mathbf{\Gamma}_j^{(d)}$ of all dimensions $d$ for the given support of the equation ; the number $j$ is given by the program; each generalized face has its own number $j$; each line of the graph contains all generalized faces $\mathbf{\Gamma}_j^{(d)}$ of the same dimension $d$, the first line contains the Newton's polyhedron $\mathbf{\Gamma}$, the next line contains all faces $\mathbf{\Gamma}_j^{(m+n-1)}$ of dimension $m+n-1$ and so on; the last line contains the empty set; if $\mathbf{\Gamma}_j^{(d)} \subset \mathbf{\Gamma}_k^{(d+1)}$, then they are connected by an arrow; in [Bruno, 2000, Ch. 1, Section 1], the "***structural diagram***" was used that is similar to the graph (see also [Conradi (et al.), 2017]);
- external normal vector $N_j$ for the each generalized face $\mathbf{\Gamma}_j^{(m+n-1)}$ for the second line of the graph;
- truncated equation $\hat{f}_j^{(d)} = 0$ by the given number $j$ of the generalized face or by a given normal vector $N_j$;

## Implementation of power geometry algorithms (3)

- normal cone of the corresponding generalized face: if the face

$$\mathbf{\Gamma}_j^{(d)} = \mathbf{\Gamma}_i^{(m+n-1)} \cap \mathbf{\Gamma}_k^{(m+n-1)} \cap \cdots \cap \mathbf{\Gamma}_l^{(m+n-1)},$$

  then the normal cone $\mathbf{U}_j^{(d)}$ is the conic hull of the external normals $N_i, N_k, \ldots, N_l$;
- power or logarithmic transformation of the original variables by a given normal $N$ of the hyperface. For this purpose, the algorithms for constructing the unimodular matrix described in [Bruno, Azimov, 2022; 2023a,b] are used.

1. Introduction

2. Basics of power geometry

3. Algorithms of power geometry and their implementations

4. The $k$–$\varepsilon$ model of evolution of turbulent bursts

5. Asymptotic forms of solutions to the system $S(2)$

6. Summary of results for the $k$–$\varepsilon$ model

## The $k$–$\varepsilon$ model of evolution of turbulent bursts (1)

According to [Kolmogorov, 1991; Bertsch (et al.), 1994; Galaktionov, 1999], the one-dimensional model is described by the system

$$\begin{cases} k_t = \left(\dfrac{k^2}{\varepsilon} k_x\right)_x - \varepsilon, \\ \varepsilon_t = \left(\dfrac{k^2}{\varepsilon} \varepsilon_x\right)_x - \gamma \dfrac{\varepsilon^2}{k}. \end{cases} \tag{6}$$

Here, time $t$ and coordinate $x$ are independent variables, the turbulent density $k$ and the dissipation rate $\varepsilon$ are dependent variables, and $\gamma$ is a real parameter. Here, $m = n = 2$, $m + n = 4$ and $x_1 = t$, $x_2 = x$, $y_1 = k$, $y_2 = \varepsilon$.

## The $k$–$\varepsilon$ model of evolution of turbulent bursts (2)

In [Bruno, Batkhin, 2022; 2023] system (6) are written as differential sums, such truncated systems are selected, which have one 3-dimensional equation, power and logarithmic transformations are applied and more simple systems are obtained. If they are not solvable, the computations are repeated till solvable systems are obtained. Their solutions, written in initial coordinates, are asymptotic forms of solutions to initial system.

According to Theorem 3 let us introduce new dependent variables:

$$u = Z^{R_1} = t^{-1} k \varepsilon^{-1}, \quad v = Z^{R_2} = x^{-2} k^3 \varepsilon^{-2}.$$

Then

$$k = \frac{x^2 v}{t^2 u^2}, \quad \varepsilon = \frac{x^2 v}{t^3 u^3}. \tag{7}$$

# The $k$–$\varepsilon$ model of evolution of turbulent bursts (3)

This is a power transformation (4) with block matrix $\alpha$, where

$$\alpha_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \alpha_{12} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \quad \alpha_{22} = \begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix}.$$

This power transformation is constructed directly on the support of the system such that it lies in the coordinate plane.

## The $k$–$\varepsilon$ model of evolution of turbulent bursts (4)

Change of the variables (7) leads the system (6) to the form

$$\begin{aligned}
&ut(\ln v)_t - 2u - 2tu_t = \\
&= v\left(6 - 12U + 7V + 6U^2 - 7UV + 2V^2 - 2xU_x + xV_x\right) - 1, \\
&ut(\ln v)_t - 3u - 3tu_t = \\
&= v\left(6 - 17U + 7V + 12U^2 - 10UV + 2V^2 - 3xU_x + xV_x\right) - \gamma,
\end{aligned} \tag{8}$$

where $U = x\left(\ln u\right)_x$, $V = x\left(\ln v\right)_x$.

Below, we assume that each intermediate variable is different from identical zero. Thus, we can consider its logarithm

## The $k$–$\varepsilon$ model of evolution of turbulent bursts (5)

After the logarithmic transformation,

$$\tau = \ln t, \quad \xi = \ln x \tag{9}$$

the system (8) takes the form

$$\begin{aligned} &u(\ln v)_\tau - 2u - 2u_\tau = \\ &= v\left(6 - 12U + 7V + 6U^2 - 7UV + 2V^2 - 2U_\xi + V_\xi\right) - 1, \end{aligned} \tag{10}$$

$$\begin{aligned} &u(\ln v)_\tau - 3u - 3u_\tau = \\ &v\left(6 - 17U + 7V + 12U^2 - 10UV + 2V^2 - 3U_\xi + V_\xi\right) - \gamma, \end{aligned} \tag{11}$$

where $U = (\ln u)_\xi$, $V = (\ln v)_\xi$.

## The $k$–$\varepsilon$ model of evolution of turbulent bursts (6)

Below, all computations are performed for the system $S$ consisting of a linear combination of the original equations:

1. Equation $E1S$ is the difference of the Equations (10) and (11);
2. Equation $E2S$ is the difference of the tripled Equation (10) and the doubled Equation (11).

As a result, the $S$ system takes the form

$$
\begin{aligned}
E1S :& u + u_\tau = 5vU - 7vU^2 + 3Uv_\xi + vU_\xi + \gamma - 1,\\
E2S :& u\,(\ln v)_\tau = 6v - 2vU + 7v_\xi - 6vU^2 + Uv_\xi + v_\xi V + \\
& v_{\xi\xi} + 2\gamma - 3.
\end{aligned}
$$

## The $k$–$\varepsilon$ model of evolution of turbulent bursts (7)

To apply the Power Geometry procedures, Equations $E1S$, $E2S$ of the $S$ system are rewritten as sums of differential monomials:

$$\begin{aligned} E1S \equiv & u^3 + (u_\tau)\, u^2 - \gamma\, u^2 - 5v\,(u_\xi)\, u - v\,(u_{\xi,\xi})\, u - 3\,(u_\xi)\,(v_\xi)\, u + \\ & + 7v\,(u_\xi)^2 + u^2 = 0, \end{aligned} \tag{12}$$

$$\begin{aligned} E2S \equiv & u^3\,(v_\tau) - (v_{\xi,\xi})\, v\, u^2 - 6v^2u^2 - 7\,(v_\xi)\, v\, u^2 - 2\gamma v\, u^2 - (v_\xi)^2\, u^2 + \\ & + 2v^2\,(u_\xi)\, u + (u_\xi)\,(v_\xi)\, vu + 6v^2\,(u_\xi)^2 + 3v\, u^2 = 0. \end{aligned} \tag{13}$$

The supports of Equations (12) and (13) are

$$\mathbf{S}(E1S) = \{[-1,0,3,0],[0,-2,2,1],[0,-1,2,1],[0,0,2,0],[0,0,3,0]\}, \tag{14}$$

$$\mathbf{S}(E2S) = \{[-1,0,3,1],[0,-2,2,2],[0,-1,2,2],[0,0,2,1],[0,0,2,2]\}. \tag{15}$$

## The $k$–$\varepsilon$ model of evolution of turbulent bursts (8)

To perform computations with a convex polyhedron of large dimension $n$, it is convenient to represent the latter as an oriented graph, all vertices of which have a unique number $j$ (identifier) and correspond to a generalized face $\mathbf{\Gamma}_j^{(d)}$ of appropriate dimension $d$.

The top vertex of the graph contains the polyhedron $\mathbf{\Gamma}$ itself, the next level contains generalized faces $\mathbf{\Gamma}_k^{(n-1)}$ of dimension $n-1$, below are generalized faces $\mathbf{\Gamma}_k^{(n-2)}$ of dimension $n-2$, and so on.

The segments connecting vertices of the graph mean that the lower element (the generalized edge) lies in the upper one (the generalized edge of higher dimension). The alternative sum of the number of vertices of the graph in the lines is equal to zero. It is the four dimensional Euler formula

# The $k$–$\varepsilon$ model of evolution of turbulent bursts (9)

The graph of the polyhedron $\mathbf{\Gamma}(E1S)$ computed by support (14) is shown in Figure 1.

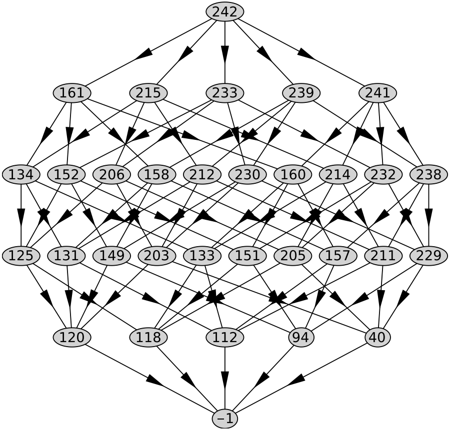


**Figure 1:** Graph of the polyhedron $\mathbf{\Gamma}(E1S)$ of (12).

## The $k$–$\varepsilon$ model of evolution of turbulent bursts (10)

The alternative sum of the numbers of elements in the rows is $1-5+10-10+5-1=0$. The polyhedron $\mathbf{\Gamma}(E1S)$ is a four-dimensional simplex and has five three-dimensional faces with identifiers $161, 215, 233, 239, 241$, computed by the program. They correspond to the external normals

$$N_{161}^{(3)} = [1,0,0,0], N_{215}^{(3)} = [-1,0,-1,0], N_{233}^{(3)} = [0,0,1,1],$$
$$N_{239}^{(3)} = [0,1,0,1], N_{241}^{(3)} = [0,-1,0,-2].$$

# The $k$–$\varepsilon$ model of evolution of turbulent bursts (11)

The graph of the polyhedron $\mathbf{\Gamma}(E2S)$ computed by support (15) is shown in Figure 2.

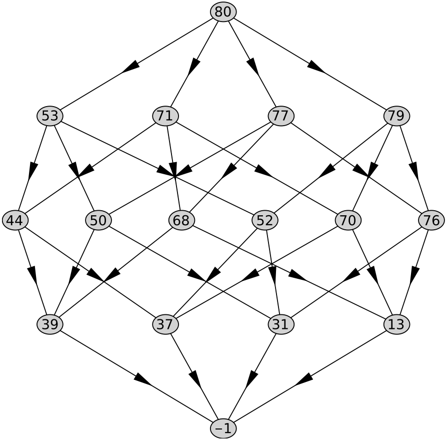


**Figure 2:** Graph of the polyhedron $\mathbf{\Gamma}(E2S)$ of (13).

## The $k$–$\varepsilon$ model of evolution of turbulent bursts (12)

The polyhedron $\mathbf{\Gamma}(E2S)$ lies in a three-dimensional plane with the normal

$$N_{80}^{(3)}(E2S) = [1,0,1,0]$$

and is a three-dimensional simplex, i.e., the Equation (13) is quasi-homogeneous.

Let us construct all truncations corresponding to the cone of problem $K[S] = \{p_1,\ p_2 \geqslant 0\}$ according to change (9). The normals $N_{161}^{(3)}$, $N_{233}^{(3)}$, $N_{239}^{(3)}$, and $N_{80}^{(3)}$ fall into the cone of problem $K[S]$. For each of the mentioned normals, we compute the truncations of the system (12), (13) and reject ***trivial***, i.e., those consisting of a single algebraic monomial.

## The $k$–$\varepsilon$ model of evolution of turbulent bursts (13)

The truncation of Equation (13) corresponding to the normal $N_{239}^{(3)}$ and the truncation of Equation (12) corresponding to the normal $N_{80}^{(3)}$ consist of one algebraic monomial $-6u^2v^2$ and $u^3$, respectively. There remain two nontrivial truncations, which we denote by $S(1)$ and $S(2)$.

The truncated system $S(1)$ depends on the variables $\xi$, $u$, $v$ and is the system of ODEs, and cone of problem $K[S(1)] = \{p_1 \geqslant 0\}$. The equations of the system have the form:

$$
\begin{aligned}
E1S(1) \equiv & u^3 - \gamma\, u^2 - 5v\,(u_\xi)\,u - v\,(u_{\xi,\xi})\,u - 3\,(u_\xi)\,(v_\xi)\,u + 7v\,(u_\xi)^2 + u^2 = 0,\\
E2S(1) \equiv & (3-2\gamma)v\,u^2 + 6v^2\,(u_\xi)^2 + (u_\xi)\,(v_\xi)\,vu + 2v^2\,(u_\xi)\,u - \\
& - (v_\xi)^2\,u^2 - 7\,(v_\xi)\,v\,u^2 - 6v^2u^2 - (v_{\xi,\xi})\,v\,u^2 = 0.
\end{aligned}
$$

## The $k$–$\varepsilon$ model of evolution of turbulent bursts (14)

The truncated system of PDEs $S(2)$ contains the variables $\tau$, $\xi$, $u$, $v$, and the cone of problem $K[S(2)] = \{p_1,\ p_2 \geqslant 0\}$. The equations of the system have the form:

$$E1S(2) \equiv (u_\tau)\, u^2 + u^3 - 5v\,(u_\xi)\, u - v\,(u_{\xi,\xi})\, u - 3\,(u_\xi)\,(v_\xi)\, u + 7v\,(u_\xi)^2 = 0, \tag{16}$$

$$\begin{aligned} E2S(2) \equiv & 6v^2\,(u_\xi)^2 + (u_\xi)\,(v_\xi)\, vu + 2v^2\,(u_\xi)\, u - (v_\xi)^2\, u^2 - 7\,(v_\xi)\, v\, u^2 - \\ & - 6v^2u^2 - (v_{\xi,\xi})\, v\, u^2 + u^3\,(v_\tau) = 0. \end{aligned} \tag{17}$$

1. Introduction
2. Basics of power geometry
3. Algorithms of power geometry and their implementations
4. The $k$–$\varepsilon$ model of evolution of turbulent bursts
5. **Asymptotic forms of solutions to the system $S(2)$**
6. Summary of results for the $k$–$\varepsilon$ model

## Asymptotic forms of solutions to system $S(2)$ (1)

Now consider the computation of the asymptotic forms of the solutions to the PDE system $S(2)$, in which Equations (16) and (17) depend on variables $\tau$, $\xi$, $u$, $v$, and cone of the problem is

$$K[S(2)] = \{p_1, p_2 \geqslant 0\}$$

## Asymptotic forms of solutions to system $S(2)$ (2)

The normal vector $N_{233}^{(3)}(E1S) = [0, 0, 1, 1]$ defines the power-logarithmic transformation

$$u = rv, \quad s = \ln v, \tag{18}$$

reducing the system $S(2)$ to the system $P(2)$ with respect to the variables $\tau$, $\xi$, $r$, and $s$ with equations:

$$\begin{aligned} E1P(2) \equiv & r^3 (s_\tau) + 3r^2 (s_\xi)^2 + r^3 + r^2 (r_\tau) - 5r^2 (s_\xi) - r^2 (s_{\xi,\xi}) + \\ & + 9r (r_\xi) - 5r (r_\xi) - r (r_{\xi,\xi}) + 7 (r_\xi)^2 = 0, \end{aligned} \tag{19}$$

$$\begin{aligned} E2P(2) \equiv & r^3 (s_\tau) + 5r^2 (s_\xi)^2 - 5r^2 (s_\xi) - r^2 (s_{\xi,\xi}) + 13r (r_\xi) (s_\xi) - 6r^2 + \\ & + 2r (r_\xi) + 6 (r_\xi)^2 = 0. \end{aligned} \tag{20}$$

## Asymptotic forms of solutions to system $S(2)$ (3)

The cone of problem of the system $P(2)$ is $K = \{p_1, p_2, p_4 \geqslant 0\}$.

The supports of Equations (19) and (20) of the system $P(2)$ are:

$$\mathbf{S}(E1P(2)) = \{[-1,0,3,0],[-1,0,3,1],[0,-2,2,0],[0,-2,2,1],\\ [0,-2,2,2],[0,-1,2,0],[0,-1,2,1],[0,0,3,0]\},\\ \mathbf{S}(E2P(2)) = \{[-1,0,3,1],[0,-2,2,0],[0,-2,2,1],[0,-2,2,2],\\ [0,-1,2,0],[0,-1,2,1],[0,0,2,0]\}$$

## Asymptotic forms of solutions to system $S(2)$ (4)

The normals to the three-dimensional faces of the convex polyhedron $\mathbf{\Gamma}(E1P(2))$ are

$$N^{(3)}_{485} = [0,-1,2,0], \quad N^{(3)}_{647} = [0,1,-1,0], \quad N^{(3)}_{701} = [-1,0,-1,0],$$
$$N^{(3)}_{707} = [0,0,0,-1], \quad N^{(3)}_{713} = [1,1,0,1], \quad N^{(3)}_{727} = [1,0,0,0].$$

The convex polyhedron $\mathbf{\Gamma}(E2P(2))$ is a three-dimensional simplex, i.e., the support of the equation $E2P(2)$ lies in the hyperplane with normals $N^{(3)}_{700} = [1{,}0{,}1{,}0]$ and $N^{(3)}_{701} = [-1{,}0,-1{,}0]$

The normals with numbers 647, 700, 713, and 727 are suitable, and we denote the corresponding systems by $S(2{,}1)$, $S(2{,}2)$, $S(2{,}3)$, and $S(2{,}4)$

# Asymptotic forms of solutions to system $S(2)$ (5)

The truncated system $S(2{,}1)$ contains the trivial truncated equation $E2S(2{,}1) \equiv -6r^2 = 0$, and the truncated system $S(2{,}2)$ contains the trivial equation $E1S(2{,}2) \equiv r^3 = 0$. Therefore, we do not consider these systems below

## Analysis of truncated system $S(2{,}3)$ (1)

The PDE system $S(2{,}3)$ corresponding to the normal $N_{713}^{(3)} = [1, 1, 0, 1]$ consists of equations:

$$E1S(2{,}3) \equiv \left(s_{\tau}\right) r + 3 \left(s_{\xi}\right)^2 - 5 s_{\xi} + r = 0,$$
$$E2S(2{,}3) \equiv \left(s_{\tau}\right) r + 5 \left(s_{\xi}\right)^2 - 5 s_{\xi} - 6 = 0,$$

derived from the corresponding equations of the system $P(2)$ after reduction by the multiplier $r^2$.

## Analysis of truncated system $S(2{,}3)$ (2)

Excluding the function $r$ from $E2S(2{,}3)$ and substituting it into $E1S(2{,}3)$, we obtain the equation:

$$E1S(2{,}3)' \equiv -2\left(s_\xi\right)^2\left(s_\tau\right) - 5\left(s_\xi\right)^2 + 5s_\xi + 6s_\tau + 6 = 0, \qquad (21)$$

which we consider as one PDE. It can be solved by the method of separation of variables, considering the required function $s(\tau,\xi)$ in the form of $s(\tau,\xi) = s_1(\tau) + s_2(\xi)$.

## Analysis of truncated system $S(2{,}3)$ (3)

Then, after substitution, it turns out that Equation (21) can be considered as the equation of an algebraic curve of genus $0$ with respect to the derivatives $(s_1)_\tau$ and $(s_2)_\xi$. This curve allows a rational parametrization

$$(s_1)_\tau = -\frac{5C_1^2 - 5C_1 - 6}{2(C_1^2 - 3)}, \quad (s_2)_\xi = C_1,$$

where $C_1$ is an arbitrary constant.

## Analysis of truncated system $S(2{,}3)$ (4)

Hence, the solution of the system $S(2{,}3)$ is the following:

$$\mathrm{Sol}S(2{,}3):\left\{ r(\tau,\xi) = 2(C_1^2-3), \quad s(\tau,\xi) = \frac{\left(5C_1^2-5C_1-6\right)\tau}{-2(C_1^2-3)} + C_1 + C_2\xi \right\}$$

which, according to (18), in the $u, v$ variables is written as

$$u = 2C_2\left(C_1^2-3\right)\mathrm{e}^w, \quad v = C_2\mathrm{e}^w,$$

where $w = \dfrac{\left(5C_1^2-5C_1-6\right)}{-2(C_1^2-3)}\tau + C_1\xi$, and $C_2$ is an arbitrary constant.

1. Introduction

2. Basics of power geometry

3. Algorithms of power geometry and their implementations

4. The $k$–$\varepsilon$ model of evolution of turbulent bursts

5. Asymptotic forms of solutions to the system $S(2)$

6. **Summary of results for the $k$–$\varepsilon$ model**

## Summary of results for the $k$–$\varepsilon$ model

In this section, we present the final results in the form of exact solutions and asymptotic forms of the solutions to the original system (6) in the initial functions $k(t,x)$ and $\varepsilon(t,x)$

## Asymptotic forms of solutions to system $S(1)$ (1)

For system $S(1)$ four groups of asymptotics were found, two of which coincided with each other

The asymptotic forms of solutions to the system $S(1,1,3)$:

$$\mathrm{Asymp}_1 S(1,1,3) : \left\{ k = \frac{\sqrt{x}\, C_2}{t^2 C_1^2}, \quad \varepsilon = \frac{\sqrt{x}\, C_2}{t^3 C_1^3} \right\},$$

$$\mathrm{Asymp}_2 S(1,1,3) : \left\{ k = \frac{C_2}{t^2 C_1^2}, \quad \varepsilon = \frac{C_2}{t^3 C_1^3} \right\},$$

$$\mathrm{Asymp}_3 S(1,1,3) : \left\{ k = \frac{x^{1/3} C_2}{t^2 C_1^2}, \quad \varepsilon = \frac{C_2}{t^3 C_1^3} \right\}.$$

## Asymptotic forms of solutions to system $S(1)$ (2)

Asymptotic forms of solutions to the system $S(1{,}1{,}4)$:

$$\mathrm{Asymp}_{1,2}S(1{,}1{,}4) : \left\{ k = \frac{x^2}{C_1 a_{1,2}^2 x^{b_{1,2}} t^2}, \quad \varepsilon = \frac{x^2}{C_1^2 a_{1,2}^3 x^{2b_{1,2}} t^3} \right\},$$

where $a_{1,2}$ and $b_{1,2}$ are given by

$$a_{1,2} = -\frac{13 \pm \sqrt{145}}{5}, \quad b_{1,2} = \frac{5 \mp \sqrt{145}}{10}.$$

## Asymptotic forms of solutions to system $S(1)$ (3)

Asymptotic forms of solutions to the system $S(1,2,1)$

$$\mathsf{Asymp}_{1,2}S(1,2,1) : \left\{ k = \frac{x^2 b_{1,2}}{t^2 C_1^2 x^{2a_{1,2}}}, \quad \varepsilon = \frac{x^2 b_{1,2}}{t^3 C_1^{3a_{1,2}}} \right\}$$

where $a_{1,2}$ and $b_{1,2}$ are given by

$$a_{1,2} = \frac{12\gamma - 17 \pm \sqrt{24\gamma + 1}}{12\gamma - 24}, \quad b_{1,2} = \gamma \pm \frac{7\sqrt{24\gamma + 1}}{12} + \frac{25}{12}.$$

The asymptotic forms of solutions to the system $S(1,2,2)$ coincide with the asymptotic forms of the system $S(1,1,3)$

## Asymptotic forms of solutions to system $S(2)$

The solution found for the truncated system $S(2{,}3)$ gives the two-parameter asymptotic form

$$\text{Asymp}S(2{,}3):\left\{k=\frac{x^{(2-C_1)}t^{(C_1-2)(C_1-3)/\left(2C_1^2-6\right)}}{4C_2\left(C_1^2-3\right)^2},\quad \varepsilon=\frac{x^{2(1-C_1)}t^{(2C_1-3)(C_1-1)/\left(C_1^2-3\right)}}{8C_2^2\left(C_1^2-3\right)^3}\right\},$$

defined for all parameter values $C_1\neq\pm\sqrt{3}$, $C_2\neq 0$

## References (1)

*Blasius H.* Grenzschichten in Flüssigkeiten mit kleiner Rebung. // Zeit. für Math. und Phys. 1908. Vol. 56. P. 1–37.

*Kolmogorov A. N.* The equation of turbulent motion of an incompressible fluid. // Selected works of A.N. Kolmogorov. I. Mathematics and Mechanics / ed. by V. M. Tikhomirov. Dordrecht: Kluwer Academic Publisher, 1991. P. 328–330.

*Bertsch M., Dal Pazo R., Kersner R.* The evolution of turbulent bursts: the b-epsilon model. // Europ. J. Appl. Math. 1994. Vol. 5. P. 537–557.

*Bruno A. D.* Algorithms of the nonlinear analysis. // Russian Math. Surveys. 1996. Vol. 51, no. 5. P. 956.

*Galaktionov V. A.* Invariant solutions of two models of evolution of turbulent bursts. // Europ. J. Appl. Math. 1999. Vol. 10. P. 237–249.

*Bruno A. D.* Power Geometry in Algebraic and Differential Equations. Amsterdam: Elsevier Science, 2000.

## References (2)


*Bruno A. D., Shadrina T. V.* Axisymmetric boundary layer on a needle. // Transactions of Moscow Math. Soc. 2007. Vol. 68. P. 201–259.

*Conradi C., Feliu E., Mincheva M., Wiuf C.* Identifying parameter regions for multistationarity. // PLOS Computational Biology / ed. by P. K. Maini. 2017. Oct. Vol. 13, no. 10. e1005751. DOI: 10.1371/journal.pcbi.1005751. URL: https://doi.org/10.1371%2Fjournal.pcbi.1005751.

*Bruno A. D., Azimov A. A.* Computation of unimodular matrices. // KIAM Preprints. Moscow, 2022. No. 36. P. 1–20. DOI: https://doi.org/10.20948/prepr-2022-46.

*Bruno A. D., Batkhin A. B.* Computation of asymptotic forms of solutions to system of nonlinear partial differential equations. // Preprints of KIAM. 2022. No. 48. P. 36. DOI: 10.20948/prepr-2022-48. (in Russian).

## References (3)

*Bruno A. D., Azimov A. A.* Computation of unimodular matrices of power transformations. // Programming and Computer Software. 2023a. Vol. 49, no. 1. P. 32–41. DOI: 10.1134/S0361768823010036.

*Bruno A. D., Azimov A. A.* Computing unimodular matrices of power transformations. // International Conference Polynomial Computer Algebra '2023'; St. Petersburg, May 17–22, 2023 / ed. by N. N. Vassiliev. St. Petersburg, Russia: VVM Publisher, 2023b. P. 27–31. ISBN 978-5-9651-1473-3.

*Bruno A. D., Batkhin A. B.* Asymptotic forms of solutions to system of nonlinear partial differential equations. // Universe. 2023. Vol. 9, no. 1. P. 35. DOI: 10.3390/universe9010035.

Conference site: `http://www.ccas.ru/ca/conference`

# 5th International Conference "Computer Algebra", Moscow, June 26–28, 2023

The 5th international Moscow conference “Computer Algebra” will be held **online** from June 26 to June 28, 2023. The conference will be co-organized by Federal Research Center “Computer Science and Control” of Russian Academy of Sciences (CCAS), Peoples’ Friendship University of Russia (PFUR) and Keldysh Institute of Applied Mathematics of Russian Academy of Sciences (KIAM RAS).

Table of Contents


- 5th International Conference "Computer Algebra", Moscow, June 26–28, 2023
  - Conference Materials
  - Location and Time
  - Conference Program
  - Main Topics
  - Important Dates
  - Program Committee
  - Organising Committee
  - Conference Template
  - Participation Fee
  - Previous Years Conference Pages


## Conference Materials

To be published – please visit later.

## Location and Time

Conference meetings will be held online. The connection details and starting time of the conference sessions are to be communicated to the participants later.

# Conference site: `http://www.ccas.ru/ca/conference`

## Main Topics

The two main topics of the conference are the following:

1. **Open questions in computer algebra**. In particular, the focus is on problems which as yet are still unsolved, but for which it is reasonable to expect that with concentrated research efforts they could be worked out in the near future.

2. **Systematization (classification) of computer-algebraic tools**. In particular, the goal is to propose principles for selection of appropriate computer-algebraic tools for solving concrete mathematical and applied problems.

Besides these two particular topics, the more traditional topics such as **symbolic algorithms, mathematical software, and application aspects** are discussed at the conference as well.

**Key words**: computer algebra, symbolic algorithms, implementation, software aspects, applied aspects, open questions, computer-algebraic tools.

## Important Dates

- Abstract submission: **due April 28, 2023** — ! The use of Conference template is required!
- Notification of acceptance: **by May 15, 2023**.

**Please send** the corresponding TeX and PDF files for the submission to email: moscowcaconf@gmail.com

Thanks for your attention!