How many roots of a random polynomial system on a compact Lie group are real?

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A finite linear combination of matrix elements of a finite dimensional real representation π of a Lie group K is said to be a real π-polynomial on K. If K is compact then any π-polynomial uniquely extends to a holomorphic function on the complexification $K_{\mathbb{C}}$ of K. For example, any trigonometric polynomial

$$f_m(\theta) = c + \sum_{1\leq k\leq m,\ \alpha_k,\beta_k\in\mathbb{R}} \alpha_k \cos(k\theta) + \beta_k \sin(k\theta)$$

on the 1-dimensional torus K (that is the unit circle $\{e^{i\theta}\colon \theta\in\mathbb{R}\}\subset\mathbb{C}$) extends uniquely to a Laurent polynomial

$$P_m(z) = c + \sum_{k\leq m,\ a_k\in\mathbb{C}} a_k z^k + \bar{a}_k z^{-k}$$

on $\mathbb{C}\setminus 0$.

For a system of n π-polynomials, where $n=\dim(K)$, we consider the proportion of real roots, that is the ratio of the number of roots in K to the number of roots in $K_{\mathbb{C}}$. The source of these calculations is the following result by M. Kac [Ka]: *the expected proportion of real zeros of a random real polynomial of degree m asymptotically equals $\frac{2}{\pi}\frac{\log m}{m}$*.

Replacing ordinary polynomials with Laurent ones (see [K1]) and then with arbitrary π-polynomials on a compact Lie group leads to an unexpected result. It turns out that for growing representation π and random system of π-polynomials, the expected proportion of real roots converges not to 0, but to a nonzero constant. The limit is calculated in terms of the volumes of some compact convex sets that determine the growth of the representation π. For a 1-dimensional torus K the limit is $1/\sqrt{3}$.

References

[Ka] M. Kac. On the average number of real roots of a random algebraic equation. Bull. Amer. Math. Soc. 49 (1943), 314–320 (Correction: Bull. Amer. Math. Soc., Volume 49, Number 12 (1943), 938–938)

[K1] B. Ya. Kazarnovskii. How many roots of a system of random trigonometric polynomials are real? Sbornik:Mathematics (213:4), 2022, 27–37 (in Russian)

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