Zeroes of π-polynomials

How many roots of a random polynomial system on a compact Lie group are real?

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Kac theorem

The following question was popular at the beginning of the 20th century: *what is the probability that the zero of polynomial of degree m with real random coefficients is real?* The answer of Mark Kac (M. Kac. On the average number of real roots of a random algebraic equation. Bull. Amer. Math. Soc. 49 (1943), 314–320; Correction: Bull. Amer. Math. Soc., Volume 49, Number 12 (1943), 938–938) was

$$\frac{2}{\pi}\frac{\log m}{m}.$$

The equality is asymptotic for large values of degree m. The polynomial coefficients are assumed to be independent normal random variables with zero expectations. The answer seems to be reasonable: non zero but very small.

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We consider the question, replacing the pair $\mathbb{R} \subset \mathbb{C}$ by the pair $K \subset K_{\mathbb{C}}$, where K is a compact Lie group and $K_{\mathbb{C}}$ is the complexification of K. Recall that

- $K_{\mathbb{C}}$ is a complex connected Lie group, $\dim_{\mathbb{C}}(K_{\mathbb{C}}) = \dim(K)$
- the Lie algebra of $K_{\mathbb{C}}$ is a complexification of the Lie algebra of K
- K is a maximal compact subgroup of $K_{\mathbb{C}}$.

Complexifications of torus $K = \{\mathrm{e}^{\mathrm{ix_1}}, \ldots, \mathrm{e}^{\mathrm{ix_n}}\}$, unitary group $U(n)$, specialy unitary group, and ... are respectively $(\mathbb{C} \setminus 0)^n$, $GL(n)$, $SL(n)$, and ...

Kac theorem for a circle

The simplest example of a compact Lie group is $K = \{\mathrm{e}^{\mathrm{ix}}\}$ the unit circle in $\mathbb{C}$. Complexification $K_{\mathbb{C}}$ is a group non zero complex numbers $\mathbb{C} \setminus 0$. By definition, the real Laurent polynomial of degree m is a polynomial of the form

$$P_m(z) = \sum_{0 \le k \le m} a_k z^k + \overline{a}_k z^{-k}$$

The restriction of P_m to the unit circle is a trigonometric polynomial. I.e.

$$P_m(\mathrm{e}^{\mathrm{ix}}) = \sum_{\mathrm{k} \le \mathrm{m}, \alpha_{\mathrm{k}}, \beta_{\mathrm{k}} \in \mathbb{R}} \alpha_{\mathrm{k}} \cos(\mathrm{kx}) + \beta_{\mathrm{k}} \sin(\mathrm{kx}).$$

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The expected number of real zeros of P_m **(not asymptotically but) exactly equals** $2\sqrt{m(m+1)/3}$.
(See in J. Angst, F. Dalmao and G. Poly. Proc. Amer. Math. Soc. (147:1), 2019, 205–214 or in arXiv:1706.01654).
So **the probability of real zero equals** $\sqrt{(m+1)/(3m)}$**, and converges to** $1/\sqrt{3}$ **as** $m \to \infty$**.** Note that since $\sqrt{3} < 2$, so **the most of zeros are real!**

π-polynomials

Let π be a finite dimensional representation of a group K. A finite linear combination of matrix elements π is said to be a π-polynomial on K. In doing so, **if the representation π is real then the coefficients also are assumed to be real**.
In a more invariant way, a π-polynomial can be defined as a linear functional on the space of representation operators.

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Example from the Kac theorem for a circle

Let us consider the plane rotation with the angle kx as a representation $r_k(\mathrm{e}^{\mathrm{ix}})$ of T^1 in $\mathbb{R}^2$, and let

$$\pi_m = \bigoplus_{k \leq m} r_k$$

be a representation of T^1 in $\mathbb{R}^{2m}$. Then the space of π_m-polynomials is the same as the space of trigonometric polynomials of the form

$$\sum_{k \leq m, \alpha_k, \beta_k \in \mathbb{R}} \alpha_k \cos(kx) + \beta_k \sin(kx)$$

Real zeros of real $\pi^{\mathbb{C}}$-polynomials

Let $K_{\mathbb{C}}$ be a complexification of the compact group K. Recall that $K_{\mathbb{C}}$ is a complex connected Lie group, such that 1) the Lie algebra of $K_{\mathbb{C}}$ is a complexification of the Lie algebra of K, and 2) K is a maximal compact subgroup of $K_{\mathbb{C}}$. Any finite dimensional representation $\pi\colon K \to \mathrm{Aut}E$ uniquely extends to the holomorphic representation $\pi^{\mathbb{C}}\colon K_{\mathbb{C}} \to \mathrm{Aut}(E \otimes_R \mathbb{C})$. So any π-polynomial on K uniquely extends to the $\pi^{\mathbb{C}}$-polynomial on $K_{\mathbb{C}}$. For real π, all these extensions are called **real $\pi^{\mathbb{C}}$-polynomials** on $K^{\mathbb{C}}$. The zero $x \in K$ of real $\pi^{\mathbb{C}}$-polynomial is called **the real zero of real $\pi^{\mathbb{C}}$-polynomial**.

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Example from the Kac theorem for a circle

Let as in the previous slide, $\pi_m = \bigoplus_{k\leq m} r_k$, The π_m-polynomials are $\sum_{k\leq m,\alpha_k,\beta_k\in\mathbb{R}} \alpha_k \cos(kx) + \beta_k \sin(kx)$. Then the space of real $\pi_m^{\mathbb{C}}$-polynomials is the same as the space of Laurent polynomials of the form

$$\sum_{0\leq k\leq m} a_k z^k + \overline{a}_k z^{-k}$$

The expected proportion of real roots

Let $\dim K = n$, and let π be a *real representation* of K. For a system of n real $\pi^{\mathbb{C}}$-polynomials, the ratio of the number of it's *common real zeros* to the number of all common zeros in $K_{\mathbb{C}}$ is said to be **the proportion of real roots**. We also define **the expected proportion of real zeros** $\mathrm{real}(\pi)$ **for a random system of real** $\pi^{\mathbb{C}}$**-polynomials**. Our goal is the calculation of asymptotics $\mathrm{real}(\pi)$ for growing representation π.

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Example from the Kac theorem for a circle

Let π_m be as in the previous slides. Then by Kac theorem for a circle, the expected proportion $\mathrm{real}(\pi_m)$ of real zeroes of real $\pi_m^{\mathbb{C}}$-polynomials equals

$$\sqrt{(m+1)/(3m)},$$

and so

$$\lim_{m\to\infty} \mathrm{real}(\pi_m) = \lim_{m\to\infty} \sqrt{(m+1)/(3m)} = 1/\sqrt{3}$$

Kac theorem for Laurent polynomials in many variables

I formulate the theorem without indicating its origin from compact torus representations (B. Ya. Kazarnovskii. How many roots of a system of random trigonometric polynomials are real? Sbornik:Math. (213:4), 2022, 27–37). This origin is analogous to the case of a 1-dimensional torus. Let B_m be a ball in $\mathbb{R}^n$ with the radius m and centre at the origin. The Laurent polynomial of degree $\leq m$

$$P(z) = \sum_{k \in \mathbb{Z}^n \cap B_m} a_k z^k,$$

is real if and only if $\forall k\colon\ a_{-k} = \overline{a_k}$.

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Let real_m be the expected proportion of real roots (i.e. the roots from T^n) for random systems of n real Laurent polynomials of degree $\leq m$. Then

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Theorem 1.

$$\lim_{m\to\infty} \mathrm{real}_m = \left(\frac{\sigma_{n-1}}{\sigma_n}\beta_n\right)^{\frac{n}{2}}$$

where

$$\beta_n = \int_{-1}^{1} x^2(1-x^2)^{\frac{n-1}{2}}\,dx$$

and σ_k is a volume of the k-dimensional unit ball.

The values of β_n

For those who are interested in the values of the constants, we give a table of $\beta_n = \int_{-1}^{1} x^2(1-x^2)^{\frac{n-1}{2}}\,dx$ for $n \le 20$.

n	1	2	3	4	5	6	7	8	9	10
β_n	$\frac{2}{3}$	$\frac{\pi}{8}$	$\frac{4}{15}$	$\frac{\pi}{16}$	$\frac{16}{105}$	$\frac{5\pi}{128}$	$\frac{32}{315}$	$\frac{7\pi}{256}$	$\frac{256}{3465}$	$\frac{21\pi}{1024}$
n	11	12	13	14	15	16	17	18	19	20
β_n	$\frac{512}{9009}$	$\frac{33\pi}{2048}$	$\frac{4096}{109395}$	$\frac{429\pi}{32768}$	$\frac{2048}{45045}$	$\frac{715\pi}{65536}$	$\frac{65536}{2078505}$	$\frac{2431\pi}{262144}$	$\frac{131072}{4849845}$	$\frac{4199\pi}{524288}$

Remark 1. If $n=1$, then $\sqrt{\frac{\sigma_0}{\sigma_1}\beta_1} = \sqrt{\frac{\beta_1}{2}} = \frac{1}{\sqrt{3}}$.

Remark 2. The expression $x^2(1-x^2)^{\frac{n-1}{2}}dx$ is a so-called Chebyshev differential binomial. In ("Sur l'integration des differentielles irrationnelles". Journal de math. pure et appl., 1853, 18, p. 87–111) Chebyshev proved that $x^m(a+bx^n)^p dx$ is not integrated by elementary functions apart from the three cases of integrability discovered by Euler. For odd n the above expression falls into the first case, and for even n it belongs to the third case.

Preliminaries from group theory

The Kac theorem deals with the growing sequence of spaces of polynomials of increasing degree m. Instead we need some growing real representation π_m of the compact group K. To construct it, we use the following description of irreducible real representations K.

- T^k, $\mathfrak{t}$ and $\mathfrak{t}^*$ are respectively the maximal torus in K, the Lie algebra of T^k and the space of linear functionals on $\mathfrak{t}$;
- $\mathbb{Z}^k \subset \mathfrak{t}^*$ is a lattice of differentials of torus characters;
- W^* is a Weyl group in the space $\mathfrak{t}^*$.

Proposition: There exists a mapping $\mathcal{W}\colon \lambda \mapsto \pi_\lambda$ of $\mathbb{Z}^k$ to the set of irreducible real representations K, such that
(1) $\mathcal{W}$ is surjective
(2) if $W^*(\lambda) = W^*(\mu)$ or $W^*(\lambda) = W^*(-\mu)$ then $\pi_\lambda = \pi_\mu$, else $\pi_\lambda \neq \pi_\mu$

Now we can define the growing representation π_m as

$$\pi_m = \sum_{\lambda \in B_m \cap \mathbb{Z}^k} \pi_\lambda,$$

where $B_m \subset \mathfrak{t}^*$ is the ball of the radius m and the centre at the origin.

Example: If $K = T^1$ then π_m is the same as in Kac theorem for a unit circle.

Kac theorem for simple Lie group

B. Kazarnovskii. How many roots ... ? https://arxiv.org/pdf/2208.14711.pdf

Here we suppose that the group K is simple, and use the coadjoint invariant metric in $\mathfrak{g}^*$, which is dual to the Killing metric in $\mathfrak{g}$. We consider the representation π_m from the previous slide.

Theorem 2. Let α, ρ, and $P(\lambda)$ be respectively the highest weight of the adjoint representation μ_α of K, the half-sum of all positive roots, and $\prod_{\beta\in R^+}(\lambda,\beta)$, where R^+ is the set of positive roots. Then

$$\lim_{m\to\infty} \mathrm{real}(\pi_m) = \frac{P^2(\rho)}{(2\pi)^n(n+2)^{n/2}(\alpha,\alpha+2\rho)^{n/2}}$$

Remark 1. The Killing product $(\alpha,\alpha+2\rho)$ equals the eigenvalue of the Casimir operator in the space μ_α-polynomials

Remark 2. The representation π_m contains irreducible components of high multiplicity, but, by definition, the space of π-polynomials does not change with increasing non-zero multiplicities of irreducible components π.

A few words about the proof, I (Newton ellipsoid)

We define the coadjointly-invariant ellipsoid $\mathrm{Ell}(\pi)$ in the space $\mathfrak{g}^*$ called the Newton ellipsoid of representation π. *If K is simple then* $\mathrm{Ell}(\pi)$ *is a ball of some radius with the centre at the origin.* Using (D. Akhiezer, B. Kazarnovskii. Average number of zeros and mixed symplectic volume of Finsler sets. Geom. Funct. Anal., (28:6), 2018, 1517–1547), we prove that **the mean number of common zeros of a random system** $f_1, \dots, f_n$ **of** n π**-polynomials equals** $\mathrm{vol}(\mathrm{Ell}(\pi))$.

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Example. Let $\pi_m \colon T^1 \to \mathrm{Aut}(\mathbb{R}^{2m})$ be, as in previous slides, a sum of irreducible representations $r_1, \dots, r_m$, where $r_k(\mathrm{e}^{\mathrm{i}x})$ is a plane rotation with the angle kx. Then the π_m-polynomials are the trigonometric polynomials of the form $P_m = \sum_{k \le m} a_k \cos(2\pi kx) + b_k \sin(2\pi kx)$, and the Newton ellipsoid is a line segment with the ends

$$\pm\sqrt{\frac{2}{2m+1}\sum_{1\le k\le m} k^2} = \pm\sqrt{\frac{m(m+1)}{3}}.$$

Hence the mean number of zeros of a random trigonometric polynomial P_m equals $2\sqrt{\frac{m(m+1)}{3}}$.

A few words about the proof, II (Newton body)

We define the compact convex set in the space $\mathfrak{g}^*$ called the Newton body $\mathcal{N}(\pi)$ of representation π. The set $\mathcal{N}(\pi)$is coadjointly invariant, that is together with any of its points contains its coadjoint orbit.
Using (B. Kazarnovskii. Newton polyhedra and the Bezout formula for matrix-valued functions of finite-dimensional representations. Funct. Anal. and Appl., (21:4), 1987, 319–321 (in Russian)), we prove that the number of common zeros of almost all systems of n $\pi^{\mathbb{C}}$-polynomials equals $\mathrm{vol}(\mathcal{N}(\pi))$. Hence, for the expected proportion of real roots we have

$$\mathrm{real}(\pi) = \frac{\mathrm{vol}(\mathrm{Ell}(\pi))}{\mathrm{vol}(\mathcal{N}(\pi))}$$

By the definition of the Newton body, for representation π_m from Theorem 2, the Newton body $\mathcal{N}(\pi_m)$ asymptotically equals the ball of radius m.

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Since the Newton ellipsoid $\mathrm{Ell}(\pi_m)$ is also a ball, then to calculate the limit of $\mathrm{real}(\pi_m)$ for $m \to \infty$ it suffices to find the asymptotics of the radius of the ball $\mathrm{Ell}(\pi_m)$ as $m \to \infty$.

This calculation is the last step of the proof.

THANKS FOR ATTENTION !