Parametric expansions of an algebraic variety near its singularities

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Talk outlook

1. Introduction
2. The Implicit Function Theorem
3. The Newton polyhedron
4. Power transformations
5. Parametric expansion of solutions
6. Variety Ω and its singularities
7. The structure of the variety Ω near the SP $P_1^{(3)}$
8. The structure of the variety Ω near the curve $\mathcal{J}$ of SP

Abstract

Now there is a method, based on Power Geometry, that allows to find asymptotic forms and asymptotic expansions of solutions to different kinds of non-linear equations near their singularities. The method contains three algorithms:

(1) Reducing equation to its normal form

(2) Separating truncated equations

(3) Power transformations of coordinates

Here we describe the method for the simplest case: a single algebraic equation, and apply it to an algebraic variety, described by an algebraic equation of order 12 in three variables. The variety was considered in study of Einstein's metrics and has several singular points and singular curves. Near some of them we compute a local parametric expansion of the variety.

1. Introduction
2. The Implicit Function Theorem
3. The Newton polyhedron
4. Power transformations
5. Parametric expansion of solutions
6. Variety Ω and its singularities
7. The structure of the variety Ω near the SP $P_1^{(3)}$
8. The structure of the variety Ω near the curve $\mathcal{J}$ of SP

Introduction (1)

Here we propose a new method for solution of a polynomial equation

$$f(x_1, \ldots, x_n) = 0$$

near its singular point. In the example we show computations of the method for a certain polynomial f and $n = 3$.

The method is used:

I The ***Newton polyhedron*** for separation of truncated equations and

II ***Power transformations*** for simplification of these equations.

Here the basic ideas of this method are explained for the simplest case: a single algebraic equation.

Introduction (2)

In Section 2 we give a generalization of Implicit Function Theorem. In Sections 3 and 4 we remind some constructions of Power Geometry [Bruno, 2000]. In Section 5 we explain a way of computation of asymptotic parametric expansions of solutions. In Section 6 we show a variety Ω and some its singularities. In Sections 7 and 8 we study the variety Ω near its singular point $P_3^{(1)} = (0{,}0{,}3/4)$ and near its singular line $\mathcal{J} = \{A_1 + A_2 + 1 = 0, A_3 = 1/2\}$ correspondingly.

1. Introduction

2. The Implicit Function Theorem

3. The Newton polyhedron

4. Power transformations

5. Parametric expansion of solutions

6. Variety Ω and its singularities

7. The structure of the variety Ω near the SP $P_1^{(3)}$

8. The structure of the variety Ω near the curve $\mathcal{J}$ of SP

The Implicit Function Theorem (1)

Let $X = (x_1, \dots, x_n)$, $Q = (q_1, \dots, q_n)$, then $X^Q = x_1^{q_1}, \dots, x_n^{q_n}$, $\|Q\| = \sum_{j=1}^{n} q_j$.

Theorem 1.

Let $f(X, \varepsilon, T) = \Sigma a_{Q,r}(T) X^Q \varepsilon^r$, where $0 \leqslant Q \in \mathbb{Z}^n$, $0 \leqslant r \in \mathbb{Z}$, the sum is finite and $a_{Q,r}(T)$ are some functions of $T = (t_1, \dots, t_m)$, besides $a_{00}(T) \equiv 0$, $a_{01}(T) \not\equiv 0$. Then the solution to the equation $f(X, \varepsilon, T) = 0$ has the form

$$\varepsilon = \Sigma b_R(T) X^R, \tag{1}$$

where $0 \leqslant R \in \mathbb{Z}^n$, $0 < \|R\|$, the coefficients $b_R(T)$ are functions on T that are polynomials from $a_{Q,r}(T)$ with $\|Q\| + r \leqslant \|R\|$ divided by $a_{01}^{2\|R\|-1}$. The expansion (1) *is unique.*

The Implicit Function Theorem (2)

This is a generalization of Theorem 1.1 of [Bruno, 2000, Ch. II] on the implicit function when the linear part $a_{01}(T) \not\equiv 0$ is non degenerate. In it, we must exclude the values of T near the zeros of the function $a_{01}(T)$.

Let $X = (x_1, \ldots, x_n) \in \mathbb{R}^n$ or $\mathbb{C}^n$, and $f(X)$ be a polynomial. A point $X = X^0$, $f(X^0) = 0$ is called ***simple*** if in it vector $(\partial f/\partial x_1, \ldots, \partial f/\partial x_n) \neq 0$.

Definition 1.

Let $\varphi(X)$ be some polynomial, $X = (x_1, \ldots, x_n)$. A point $X = X^0$ of the set $\varphi(X) = 0$ is called ***singular point of the k-order (SP)***, if all partial derivatives of the polynomial $\varphi(X)$ for the $x_1, \ldots, x_n$ turn into zero at this point, up to and including k-th order derivatives, and at least one partial derivative of order $k+1$ is nonzero.

1. Introduction

2. The Implicit Function Theorem

3. **The Newton polyhedron**

4. Power transformations

5. Parametric expansion of solutions

6. Variety Ω and its singularities

7. The structure of the variety Ω near the SP $P_1^{(3)}$

8. The structure of the variety Ω near the curve $\mathcal{J}$ of SP

The Newton polyhedron (1)

Let the point $X^0 = 0$ be singular. Write the polynomial in the form $f(X) = \Sigma a_Q X^Q$, where $a_Q = \text{const} \in \mathbb{R}$, or $\mathbb{C}$. Let $\mathbf{S}(f) = \{Q : a_Q \neq 0\} \subset \mathbb{R}^n$.

The set $\mathbf{S}$ is called the ***support*** of the polynomial $f(X)$. Let it consist of points $Q_1, \dots, Q_k$. The convex hull of the support $\mathbf{S}(f)$ is the set

$$\Gamma(f) = \left\{ Q = \sum_{j=1}^{k} \mu_j Q_j, \quad \mu_j \geqslant 0, \quad \sum_{j=1}^{k} \mu_j = 1 \right\},$$

which is called the ***Newton polyhedron***.

The Newton polyhedron (2)

Its boundary $\partial\Gamma(f)$ consists of generalized faces $\Gamma_j^{(d)}$, where d is its dimension of $0 \leqslant d \leqslant n-1$ and j is its number. Numbering is unique for all dimensions d.

Each (generalized) face $\Gamma_j^{(d)}$ corresponds to its:

- ***boundary subset*** $\mathbf{S}_j^{(d)} = \mathbf{S} \cap \Gamma_j^{(d)}$,
- ***truncated polynomial*** $\hat{f}_j^{(d)}(X) = \Sigma a_Q X^Q$ over $Q \in \mathbf{S}_j^{(d)}$, and
- ***normal cone***

$$\mathbf{U}_j^{(d)} = \left\{ P : \langle P, Q' \rangle = \langle P, Q'' \rangle > \langle P, Q''' \rangle, Q', Q'' \in \mathbf{S}_j^{(d)}, Q''' \in \mathbf{S} \backslash \mathbf{S}_j^{(d)} \right\},$$

where $P = (p_1, \ldots, p_n) \in \mathbb{R}_*^n$, the space $\mathbb{R}_*^n$ is conjugate (dual) to the space $\mathbb{R}^n$ and $\langle P, Q \rangle = p_1 q_1 + \ldots + p_n q_n$ is the scalar product.

The Newton polyhedron (3)

At $X \to 0$ solutions to the full equation $f(X) = 0$ tend to non-trivial solutions of those truncated equations $\hat{f}_j^{(d)}(X) = 0$ whose normal cone $\mathbf{U}_j^{(d)}$ intersects with the negative orthant $P \leqslant 0$ in $\mathbb{R}_*^n$.

1. Introduction

2. The Implicit Function Theorem

3. The Newton polyhedron

4. Power transformations

5. Parametric expansion of solutions

6. Variety Ω and its singularities

7. The structure of the variety Ω near the SP $P_1^{(3)}$

8. The structure of the variety Ω near the curve $\mathcal{J}$ of SP

Power transformations (1)

Let $\ln X = (\ln x_1, \ldots, \ln x_n)$. The linear transformation of the logarithms of the coordinates

$$(\ln y_1, \ldots, \ln y_n) \stackrel{\text{def}}{=} \ln Y = (\ln X)\alpha, \qquad (2)$$

where α is a nondegenerate square n-matrix, is called ***power transformation***.

By the power transformation (2), the monomial X^Q tranforms into the monomial Y^R, where $R = Q(\alpha^*)^{-1}$ and the asterisk indicates a transposition.

A matrix α is called ***unimodular*** if all its elements are integers and $\det \alpha = \pm 1$. For an unimodular matrix α, its inverse α^{-1} and transpose α^* are also unimodular.

Theorem 2.

For the face $\Gamma_j^{(d)}$ *there exists a power transformation* (2) *with the unimodular matrix* α *which reduces the truncated sum* $\hat{f}_j^{(d)}(X)$ *to the sum from* d *coordinates, that is,* $\hat{f}_j^{(d)}(X) = Y^S \hat{g}_j^{(d)}(Y)$ *where* $\hat{g}_j^{(d)}(Y) \equiv \hat{g}_j^{(d)}(y_1, \dots, y_d)$ *is a polynomial. Here* $S \in \mathbb{Z}^n$*. The additional coordinates* $y_{d+1}, \dots, y_n$ *are local (small).*

The article [Bruno, Azimov, 2023] specifies an algorithm for computing the unimodular matrix α of Theorem 2.

1. Introduction

2. The Implicit Function Theorem

3. The Newton polyhedron

4. Power transformations

5. Parametric expansion of solutions

6. Variety Ω and its singularities

7. The structure of the variety Ω near the SP $P_1^{(3)}$

8. The structure of the variety Ω near the curve $\mathcal{J}$ of SP

Parametric expansion of solutions (1)

Let $\Gamma_j^{(d)}$ be a face of the Newton polyhedron $\Gamma(f)$. Let the full equation $f(X) = 0$ is changed into the equation $g(Y) = 0$ after the power transformation of Theorem 2. Thus $\hat{g}_j^{(d)}(y_1, \ldots, y_d) = g(y_1, \ldots, y_d, 0, \ldots, 0)$.

Let the polynomial $\hat{g}_j^{(d)}$ be the product of several irreducible polynomials

$$\hat{g}_j^{(d)} = \prod_{k=1}^{m} h_k^{l_k}(y_1, \ldots, y_d), \tag{3}$$

where $0 < l_k \in \mathbb{Z}$. Let the polynomial h_k be one of them.

Three cases are possible:

Parametric expansion of solutions (2)

Case 1

The equation $h_k = 0$ has a polynomial solution $y_d = \varphi(y_1, \ldots, y_{d-1})$. Then in the full polynomial $g(Y)$ let us substitute the coordinates $y_d = \varphi + z_d$, for the resulting polynomial $h(y_1, \ldots, y_{d-1}, z_d, y_{d+1} \ldots, y_n)$ again construct the Newton polyhedron, separate the truncated polynomials, etc. Such calculations were made in [Bruno, Batkhin, 2012] and were shown in Introduction to [Bruno, 2000].

Case 2

The equation $h_k = 0$ has no polynomial solution, but has a parametrization of solutions $y_j = \varphi_j(T), j = 1, \ldots, d, \quad T = (t_1, \ldots, t_{d-1})$.

Parametric expansion of solutions (3)

Then in the full polynomial $g(Y)$ we substitute the coordinates

$$y_j = \varphi_i(T) + \beta_j \varepsilon, \quad j = 1, \ldots, d, \tag{4}$$

where $\beta_j = \mathrm{const}$, $\Sigma\,|\beta_j| \neq 0$, and from the full polynomial $g(Y)$ we get the polynomial

$$h = \Sigma a_{Q'',r}(T) {Y''}^{Q''} \varepsilon^r, \tag{5}$$

where $Y'' = (y_{d+1}, \ldots, y_n)$, $0 \leqslant Q'' = (q_{d+1}, \ldots, q_n) \in \mathbb{Z}^{n-d}$, $0 \leqslant r \in \mathbb{Z}$. Thus $a_{00}(T) \equiv 0$, $a_{01}(T) = \sum_{j=1}^{d} \beta_j \partial \hat{g}_j^{(d)} / \partial y_j(T)$.

Parametric expansion of solutions (4)

If in the expansion (3) $l_k = 1$, then $a_{01} \not\equiv 0$. By Theorem 1, all solutions to the equation $h_k = 0$ have the form $\varepsilon = \Sigma b_{Q''}(T) Y''^{Q''}$, i.e., according to (4) the solutions to the equation $g = 0$ have the form $y_j = \varphi_j(T) + \beta_j \Sigma b_{Q''}(T) Y''^{Q''}$, $j = 1, \dots, d$. Such calculations were proposed in [Bruno, 2018].

If in (3) $l_k > 1$, then in (5) $a_{01}(T) \equiv 0$ and for the polynomial (5) from Y'', ε we construct the Newton polyhedron by support $\mathbf{S}(h) = \left\{Q'', r : a_{Q'',r}(T) \not\equiv 0\right\}$, separate the truncations and so on.

Case 3

The equation $h_k = 0$ has neither a polynomial solution nor a parametric one. Then, using Hadamard's polyhedron [Bruno, 2018], one can compute a piecewise approximate parametric solution to the equation $h_k = 0$ and look for an approximate parametric expansion.

Parametric expansion of solutions (5)

Similarly, one can study the position of an algebraic manifold in infinity.

1. Introduction
2. The Implicit Function Theorem
3. The Newton polyhedron
4. Power transformations
5. Parametric expansion of solutions
6. **Variety Ω and its singularities**
7. The structure of the variety Ω near the SP $P_1^{(3)}$
8. The structure of the variety Ω near the curve $\mathcal{J}$ of SP

Variety Ω and its singularities (1)

The Ricci flows describe the evolution of Einstein's metrics on a variety and have 3 real parameters a_1, a_2, a_3.

The Ricci flow has a degenerate stationary point iff its parameters a_1, a_2, a_3 satisfies the equation

$$Q(s_1, s_2, s_3) \equiv (2s_1 + 4s_3 - 1)\left(64s_1^5 - 64s_1^4 + 8s_1^3 + 240s_1^2 s_3 - 1536 s_1 s_3^2 - \right.$$
$$\left. -4096 s_3^3 + 12 s_1^2 - 240 s_1 s_3 + 768 s_3^2 - 6s_1 + 60 s_3 + 1\right) - 8 s_1 s_2 (2s_1 + 4s_3 - 1)$$
$$\times (2s_1 - 32 s_3 - 1)(10 s_1 + 32 s_3 - 5) - 16 s_1^2 s_2^2 \left(52 s_1^2 + 640 s_1 s_3 + 1024 s_3^2 - 52s\right.$$
$$\left. -320 s_3 + 13\right) + 64(2s_1 - 1) s_2^3 (2s_1 - 32 s_3 - 1) + 2048 s_1 (2s_1 - 1) s_2^4 = 0,$$

where s_1, s_2, s_3 are elementary symmetric polynomials, equal respectively to $s_1 = a_1 + a_2 + a_3$, $s_2 = a_1 a_2 + a_1 a_3 + a_2 a_3$, $s_3 = a_1 a_2 a_3$.

Variety Ω and its singularities (2)

We denote the set of solutions to equation $Q(\mathbf{s}) = 0$ as variety Ω [Nikonorov, 2016].

In [Bruno, Batkhin, 2015], for symmetry reasons, the coordinates $\mathbf{a} = (a_1, a_2, a_3)$ were changed to the coordinates $\mathbf{A} = (A_1, A_2, A_3)$ by a linear transformation

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = M \cdot \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}, \; M = \begin{pmatrix} (1+\sqrt{3})/6 & (1-\sqrt{3})/6 & 1/3 \\ (1-\sqrt{3})/6 & (1+\sqrt{3})/6 & 1/3 \\ -1/3 & -1/3 & 1/3 \end{pmatrix}$$

In [Bruno, Batkhin, 2015] all SPs of the variety Ω in coordinates $\mathbf{A} = (A_1, A_2, A_3)$ were found. There are five points of the third order. Among them $P_1^{(3)} = (0{,}0{,}3/4)$. There are three second-order points and three algebraic curves of SPs of the first order. Among them is $\mathcal{J} = \{A_1 + A_2 + 1 = 0, A_3 = 1/2\}$.

Variety Ω and its singularities (3)

Below we will consider the variety Ω in the neighborhood of point $P_1^{(3)}$ and curve $\mathcal{J}$. The methods proposed in [Bruno, 2018] and described in Sections 2-5 are implemented.

1. Introduction

2. The Implicit Function Theorem

3. The Newton polyhedron

4. Power transformations

5. Parametric expansion of solutions

6. Variety Ω and its singularities

7. The structure of the variety Ω near the SP $P_1^{(3)}$

8. The structure of the variety Ω near the curve $\mathcal{J}$ of SP

Structure of Ω near the SP $P_1^{(3)}$ (1)

Near the point $P_1^{(3)}$ let us introduce the local coordinates x_1, x_2, x_3: $A_1 = x_1$, $A_2 = x_2$, $A_3 = x_3 + 3/4$ and from the polynomial $R(\mathbf{A})$ we get a polynomial of degree 12 $S_1(x_1, x_2, x_3) = R(\mathbf{A}) = Q(s_1, s_2, s_3)$. We calculate its support, the Newton polyhedron Γ_1, its faces $\Gamma_j^{(2)}$ and their external normals, using the `PolyhedralSets` package of the Maple 2021 computer algebra system [Thompson, 2016]. We get 5 faces $\Gamma_j^{(2)}$. The graph of the polyhedron Γ_1 is shown in Fig. 1.

Structure of Ω near the SP $P_1^{(3)}$ (2)

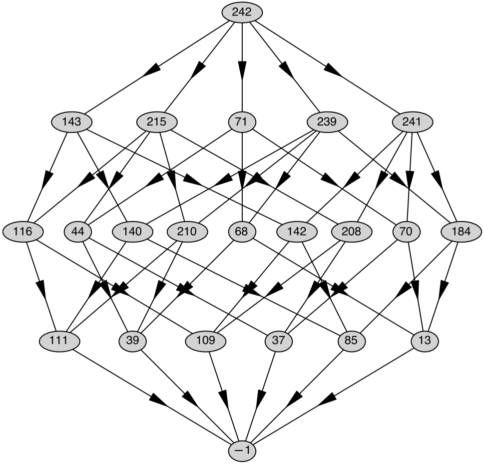

Figure 1: Graph of the polyhedron Γ_1.

Structure of Ω near the SP $P_1^{(3)}$ (3)

Each generalized face $\Gamma_j^{(d)}$ is presented by its number j in an oval. Numbers j are given by the program automatically. The top line of Fig. 1 contains the whole polyhedron Γ, the next line contains all the two-dimensional faces $\Gamma_j^{(2)}$.

Below that, the edges $\Gamma_j^{(1)}$, then the vertices $\Gamma_j^{(0)}$, and at the bottom, the empty set. A face $\Gamma_j^{(d)}$ is connected with a face $\Gamma_k^{(d+1)}$ by arrow, iff $\Gamma_j^{(d)} \subset \Gamma_k^{(d+1)}$.

The external normals to its two-dimensional faces $\Gamma_j^{(2)}$ are

$$N_{71} = (-1,-1,-1/2),\ N_{143} = (1{,}1{,}1),\ N_{215} = (-1{,}0{,}0),$$
$$N_{239} = (0,-1{,}0),\ N_{241} = (0{,}0,-1).$$

Structure of Ω near the SP $P_1^{(3)}$ (4)

The neighborhood of the point $x_1 = x_2 = x_3 = 0$ is approximately described by the truncated equation

$$\hat{f}_1 \equiv -\frac{4096}{81}81x_3^8 + \frac{3}{4}x_1^4 + \frac{3}{4}x_2^4 + \frac{64}{3}x_1^2x_3^4 - \frac{16}{3}x_1^3x_3^2 + \\ + \frac{64}{3}x_2^2x_3^4 - \frac{16}{3}x_2^3x_3^2 + \frac{3}{2}x_1^2x_2^2 + 16x_1^2x_2x_3^2 + 16x_1x_2^2x_3^2 = 0,$$

corresponding to the face $\Gamma_j^{(2)}$ of number $j = 71$ with the normal $N_{71} = (-2, -2, -1)$, which has all coordinates negative.

According to the article [Bruno, Azimov, 2023], we find the unimodular matrix $\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -2 & 1 \end{pmatrix}$ such that $N_{71}\alpha = (0,0,-1)$.

Structure of Ω near the SP $P_1^{(3)}$ (5)

Consequently, we have to do the power transformation

$$(\ln y_1, \ln y_2, \ln y_3) = (\ln x_1, \ln x_2, \ln x_3) \cdot \alpha,$$

i.e.

$$(\ln x_1, \ln x_2, \ln x_3) = (\ln y_1, \ln y_2, \ln y_3) \cdot \alpha^{-1}$$

Since $\alpha^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}$, then

$$x_1 = y_1 y_3^2, \quad x_2 = y_2 y_3^2, \quad x_3 = y_3. \tag{6}$$

Structure of Ω near the SP $P_1^{(3)}$ (6)

In this case, $\hat{f}_1(x_1, x_2, x_3) = y_3^8 \cdot F_1(y_1, y_2)$;

$$\begin{aligned} F_1(y_1, y_2) = &-\frac{4096}{81} + \frac{3}{4}y_1^4 + \frac{3}{4}y_2^4 + \frac{64}{3}y_1^2 - \frac{16}{3}y_1^3 + \frac{64}{3}y_2^2 - \\ &- \frac{16}{3}y_2^3 + \frac{3}{2}y_1^2 y_2^2 + 16 y_1^2 y_2 + 16 y_1 y_2^2. \end{aligned} \tag{7}$$

According to the `algcurves` package from the computer algebra system `Maple`, the curve $F_1(y_1, y_2) = 0$ has genus 0, parametrization

Structure of Ω near the SP $P_1^{(3)}$ (7)

$$
\begin{aligned}
&y_1 = b_1(t) = \\
&-8(21434756829626557083983t^4 + 1417074727891594177202560t^3 + \\
&+ 3170603819337258046158870 6t^2 + 3357262000619582274487921 84t + \\
&+ 833310342734734538 4379)/\delta, \\
&y_2 = b_2(t) = \\
&-56(30534309009669314405 69t^4 + 19840750299173693831 6080t^3 + \\
&+ 38835332085532533132581 58t^2 + 91935591048204912797158 48t - \\
&-26222628222183337506658650323)/\delta, \\
&\delta = 9\left(85576987369t^2 + 3099727166140t + 37630556816821\right)^2,
\end{aligned}
\tag{8}
$$

and the plot shown in Fig. 2.

Structure of Ω near the SP $P_1^{(3)}$ (8)

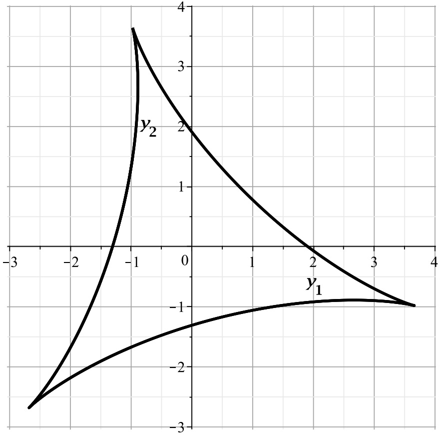

Figure 2: Plot of the curve $F_1(y_1, y_2) = 0$.

Structure of Ω near the SP $P_1^{(3)}$ (9)

This is a curvilinear triangle with vertices
$(y_1, y_2) = -\frac{8}{3}(1,1)$, $-\frac{8}{3}\left(-\frac{1+\sqrt{3}}{2}, \frac{\sqrt{3}-1}{2}\right)$, $-\frac{8}{3}\left(\frac{\sqrt{3}-1}{2}, -\frac{\sqrt{3}+1}{2}\right)$.

Now, to describe the structure of the variety Ω near the point $P_1^{(3)}$, we substitute power transformation (6) into the polynomial $S_1(x_1, x_2, x_3)$ and get the polynomial $T(y_1, y_2, y_3) = S_1/y_3^8$. It decomposes into the sum

$$T(y_1, y_2, y_3) = \sum_{k=0}^{m} T_k(y_1, y_2) y_3^k$$

with $T_0(y_1, y_2) = F_1(y_1, y_2)$ and using the command `coeff(T,y[k],m)` in CAS Maple selecting monomials containing factor x_k^m, for $k = 3$ and $m = 1$ we obtain

Structure of Ω near the SP $P_1^{(3)}$ (10)

$$T_1 \stackrel{\text{def}}{=} G(y_1, y_2) = 8y_1^4 + 16y_1^2y_2^2 + 8y_2^4 - \frac{1216}{27}y_1^3 + \frac{1216}{9}y_1^2y_2 + \\ + \frac{1216}{9}y_1y_2^2 - \frac{1216}{27}y_2^3 + \frac{3584}{27}y_1^2 + \frac{3584}{27}y_2^2 - \frac{65536}{729} \quad (9)$$

In the polynomials $T_k(y_1, y_2)$ we do the substitution

$$y_1 = b_1(t) + \varepsilon, \quad y_2 = b_2(t) + \varepsilon. \quad (10)$$

Structure of Ω near the SP $P_1^{(3)}$ (11)

We obtain a polynomial $u(\varepsilon, y_3) = T(y_1, y_2, y_3)$ with coefficients depending on t through $b_1(t)$ and $b_2(t)$. In this polynomial

$$u(\varepsilon, y_3) = \sum_{k=0}^{m} T_k(b_1 + \varepsilon, b_2 + \varepsilon) y_3^k = \sum_{p,q \geqslant 0} u_{pq} \varepsilon^p y_3^q,$$

where $u_{00} = F_1(b_1(t), b_2(t))$ of (7) so $u_{00} \equiv 0$,

$$u_{10} = \frac{\partial F_1(y_1, y_2)}{\partial y_1} + \frac{\partial F_1(y_1, y_2)}{\partial y_2} = \\ 3y_1^3 + 128/3y_1 + 3y_1 y_2^2 + 3y_1^2 y_2 + 64 y_1 y_2 + 3y_2^3 + 128/3y_2 \\ \stackrel{\text{def}}{=} H(y_1, y_2), \tag{11}$$

and in general

$$u_{pq} = \sum_{p_1+p_2=p\geqslant 1} \frac{1}{p_1!p_2!} \cdot \frac{\partial^p T_q}{\partial y_1^{p_1} \cdot \partial y_2^{p_2}} \qquad (12)$$

when $p_1, p_2 \geqslant 0, p \geqslant 1$, $y_i = b_i(t)$, $i = 1{,}2$, according to (8) and substitution (10).

Structure of Ω near the SP $P_1^{(3)}$ (13)

Now according to (9) and (11)

$$u_{10}(t) = H(b_1(t), b_2(t)) =$$
$$-32768(254517259607t^2 + 8638940893220t + 63662194408079)^3 \times$$
$$\times (23525t + 3508186)^4/(243\gamma^5),$$
$$u_{01}(t) = G(b_1(t), b_2(t)) =$$
$$5242880(23525t + 3508186)^4 \times$$
$$\times (254517259607t^2 + 8638940893220t + 63662194408079)^4/$$
$$/(19683\gamma^6),$$
$$\gamma = 85576987369t^2 + 3099727166140t + 37630556816821.$$

Structure of Ω near the SP $P_1^{(3)}$ (14)

The functions $u_{10}(t)$ and $u_{01}(t)$ each have three multiple roots

$$t_1 = -\frac{3508186}{23525}, t_{2,3} = -\frac{4319470446610}{254517259607} \pm \frac{904562081493\sqrt{3}}{254517259607}. \quad (13)$$

These values correspond to the vertices of the curvilinear triangle of Fig. 2

Structure of Ω near the SP $P_1^{(3)}$ (15)

According to Theorem 1 on the implicit function, the equation $u(\varepsilon, y_3) = 0$ has the solution as the power series over y_3

$$\varepsilon = \sum_{k=1}^{\infty} c_k(t) \cdot y_3^k, \tag{14}$$

where $c_k(t)$ are rational functions that are expressed via the coefficients $u_{pq}(t)$, which in turn are expressed via $b_1(t)$ and $b_2(t)$ according to (12).

Structure of Ω near the SP $P_1^{(3)}$ (16)

This decomposition is valid for all values of t, except maybe the roots (13). In particular,

$$c_1(t) = -\frac{u_{01}}{u_{10}} = -\frac{G}{H} = \frac{160(254517259607t^2 + 8638940893220t + 63662194408079)}{81(85576987369t^2 + 3099727166140t + 37630556816821)},$$

where the denominator has no real roots. According to (14) approximate $r \approx c_1(t)y_3$.

Structure of Ω near the SP $P_1^{(3)}$ (17)

Let us return to the initial coordinates, which for small $|y_3|$ on variety Ω are approximated by

$$x_1 = (b_1(t) + c_1(t)y_3)y_3^2, \quad x_2 = (b_2(t) + c_1(t)y_3)y_3^2. \qquad (15)$$

If $y_3 = -1/50$, i.e., $A_3 = 73/100$, the curve (15) is shown in Fig. 3.

It is similar to the curve of Fig. 11 in [Bruno, Batkhin, 2015] with $A_3 = 5/8$ near the origin.

Structure of Ω near the SP $P_1^{(3)}$ (18)

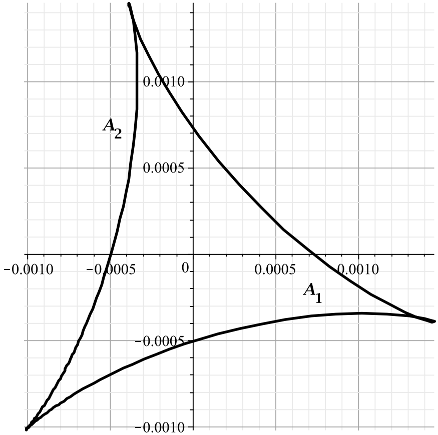

Figure 3: Curve (15) for $y_3 = -1/50$.

Structure of Ω near the SP $P_1^{(3)}$ (19)

If $y_3 = 1/20$, i.e., $A_3 = 4/5$, it is shown in Fig. 4 and is similar to the curve of Fig. 9 in [Bruno, Batkhin, 2015] with $A_3 = 1$ near the origin.

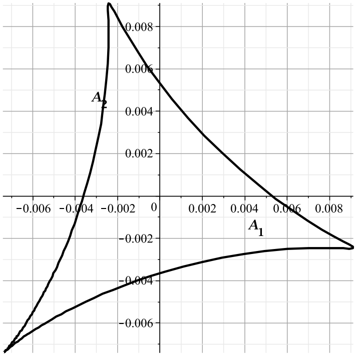

Figure 4: Curve (15) for $y_3 = 1/20$

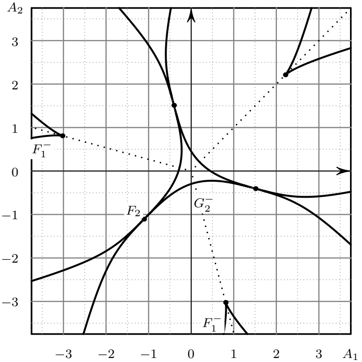

Figure 5: Fig. 9 in [Bruno, Batkhin, 2015]

1. Introduction

2. The Implicit Function Theorem

3. The Newton polyhedron

4. Power transformations

5. Parametric expansion of solutions

6. Variety Ω and its singularities

7. The structure of the variety Ω near the SP $P_1^{(3)}$

8. The structure of the variety Ω near the curve $\mathcal{J}$ of SP

Structure of Ω near the curve $\mathcal{J}$ of SP (1)

On the curve $\mathcal{J}$ and near it, let us introduce the local coordinates x_1, x_2, x_3 :

$$A_1 = x_1 - x_2 - \frac{1}{2},\ A_2 = x_1 + x_2 - \frac{1}{2},\ A_3 = \frac{1}{2} + x_3.$$

On the line $\mathcal{J}$ the coordinates $x_1 = x_3 = 0$ and x_2 is arbitrary.

Structure of Ω near the curve $\mathcal{J}$ of SP (2)

From the polynomial $R(\mathbf{A})$ we get a polynomial of degree 12

$$S_3(x_1, x_2, x_3) = R(\mathbf{A}) = Q(s_1, s_2, s_3),$$

we compute its support, the Newton polyhedron Γ_3, its faces $\Gamma_j^{(2)}$ and their external normals, using the `PolyhedralSets` package of the CAS `Maple` 2021. We obtain 7 faces $\Gamma_j^{(2)}$. The graph of the polyhedron Γ_3 is shown in Fig. 6.

Structure of Ω near the curve $\mathcal{J}$ of SP (3)

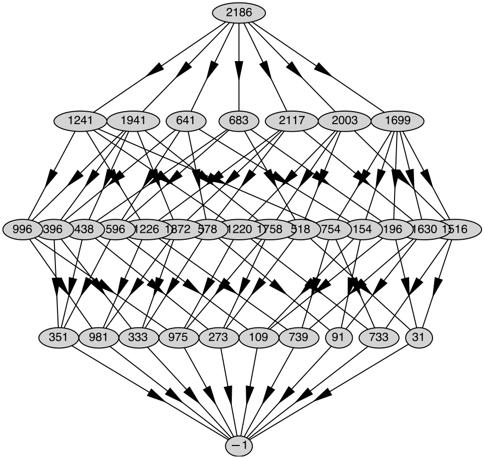

Figure 6: The graph of the polyhedron Γ_3.

Structure of Ω near the curve $\mathcal{J}$ of SP (4)

The external normals of its two-dimensional faces are $N_{641} = (-1{,}0,-1)$, $N_{683} = (-1,-1,-2)$, $N_{1241} = (1{,}1{,}1)$, $N_{1699} = (0{,}0,-1)$, $N_{1941} = (-1{,}0{,}0)$, $N_{2003} = (0,-1{,}0)$, $N_{2117} = (0{,}1{,}0)$.

The neighborhood of the line $x_1 = x_3 = 0$ is approximately described by the zeros of the truncated polynomial

$$\hat{f}_1 = -\frac{1024}{81}x_1^2x_2^4 - \frac{16384}{729}x_2^8x_1^2 + \frac{8192}{729}x_2^8x_3^2 + \frac{8192}{243}x_1^2x_2^6 + \frac{1664}{81}x_2^4x_3^2 - \\ - \frac{16}{3}x_2^2x_3^2 - \frac{6400}{243}x_2^6x_3^2 + \frac{4096}{243}x_1x_2^6x_3 - \frac{8192}{729}x_2^8x_1x_3 - \frac{512}{81}x_1x_2^4x_3,$$

corresponding to face 641 with normal $N_{641} = (-1{,}0,-1)$, which has two coordinates negative.

Structure of Ω near the curve $\mathcal{J}$ of SP (5)

According to the paper [Bruno, Azimov, 2023] we find the unimodular matrix

$$\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

such that

$$N\alpha = (0{,}0, -1)\,.$$

Hence we have to perform the power transformation

$$(\ln y_1, \ln y_2, \ln y_3) = (\ln x_1, \ln x_2, \ln x_3) \cdot \alpha,$$

i.e.

$$(\ln x_1, \ln x_2, \ln x_3) = (\ln y_1, \ln y_2, \ln y_3) \cdot \alpha^{-1}.$$

Structure of Ω near the curve $\mathcal{J}$ of SP (6)

Since $\alpha^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, then

$$x_1 = y_1 y_3,\ x_2 = y_2,\ x_3 = y_3. \tag{16}$$

In this case

$$\hat{f}_1 \left(x_1, x_2, x_3\right) = y_3^2 \cdot F_3 \left(y_1, y_2\right);$$

$$F_3 \left(y_1, y_2\right) = -\frac{16 y_2^2 \left(4 y_2^2 - 3\right)^2 \left(64 y_2^2 y_1^2 + 32 y_1 y_2^2 - 32 y_2^2 + 27\right)}{729}.$$

Structure of Ω near the curve $\mathcal{J}$ of SP (7)

The equation $F_3(y_1, y_2) = 0$ has three solutions:

1. $y_2 = 0$. It corresponds to the point $P_3^{(2)}$.
2. $y_2 = \pm\sqrt{3}/2$. It corresponds to points $P_4^{(3)}$ and $P_5^{(3)}$, which we will study separately.
3. Curve

$$\Phi(y_1, y_2) \stackrel{\text{def}}{=} 64_2^2 y_1^2 + 32 y_1 y_2^2 - 32 y_2^2 + 27 = 0.$$

Structure of Ω near the curve $\mathcal{J}$ of SP (8)

According to the procedure `genus` from the package `algcurves` program from the CAS Maple, the curve $\Phi(y_1, y_2) = 0$ has genus 0, parameterization

$$y_1 = b_1(t) \stackrel{\text{def}}{=} \frac{5t^2 + 2t - 1}{19t^2 + 22t + 7}, \quad y_2 = b_2(t) \stackrel{\text{def}}{=} -\frac{19t^2 + 22t + 7}{16t^2 + 24t + 8}, \tag{17}$$

and the graph shown in Fig. 7.

Structure of Ω near the curve $\mathcal{J}$ of SP (9)

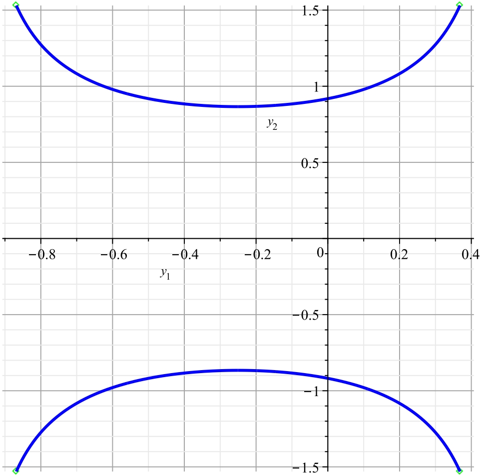

Figure 7: The curve $\Phi\left(y_1, y_2\right) = 0$.

Structure of Ω near the curve $\mathcal{J}$ of SP (10)

This curve is located in the band $-1 < y_1 < \frac{1}{2}$, it is symmetric relative to the axis $y_2 = 0$ and the vertical $y_1 = -\frac{1}{4}$. When $y_1 = -\dfrac{1}{4}$, on it $y_2 = \pm\dfrac{\sqrt{3}}{2} = \pm 0.8660254$, $t = (-5 \mp 2\sqrt{3})/13$ (i.e. on the curve $t = -0.651084$ and $t = -0.118146$). In this $|y_2| \geqslant \sqrt{3}/2$. At $y_1 = -1$, $t = -1/2$, at $y_1 = 1/2$, $t = -1$, and $y_2 = \pm\infty$.

Now to describe the structure of variety Ω near the line $\mathcal{J}$ we substitute (16) into the polynomial $S_3(\mathbf{x})$ and get the polynomial $T\,(y_1, y_2, y_3)$. It splits into the sum

$$T\,(y_1, y_2, y_3) = y_3^2 \sum_{k=0}^{m} T_k(y_1, y_2) y_3^k$$

with $T_0\,(y_1, y_2) = F_3\,(y_1, y_2)$ and using the `coeff` command we get

Structure of Ω near the curve $\mathcal{J}$ of SP (11)

$$T_1 \stackrel{\text{def}}{=} G\left(y_1, y_2\right) = \frac{16384}{243} y_1^3 y_2^4 - \frac{32768}{243} y_2^6 y_1^3 + \frac{131072}{2187} y_1^3 y_2^8 - \frac{11776}{81} y_1 y_2^4 + \\ + \frac{8192}{81} y_1 y_2^6 + \frac{1280}{27} y_2^2 y_1^2 + \frac{65536}{729} y_2^8 y_1^2 + \frac{1408}{27} y_1 y_2^2 - \frac{4096}{81} y_1^2 y_2^6 - \\ - \frac{2048}{27} y_1^2 y_2^4 + 16 + \frac{4096}{243} y_2^6 - \frac{65536}{2187} y_2^8 + \frac{13312}{243} y_2^4. \quad (18)$$

Structure of Ω near the curve $\mathcal{J}$ of SP (12)

In the polynomials $T_k(y_1, y_2)$ we substitute

$$y_1 = b_1(t) + \varepsilon, \quad y_2 = b_2(t). \tag{19}$$

We get a polynomial $u(\varepsilon, y_3) = T(y_1, y_2, y_3)/y_3^2$ with coefficients depending on t through $b_1(t)$ and $b_2(t)$. In this polynomial

$$u(\varepsilon, y_3) = \sum_{k=0}^{m} T_k(b_1 + \varepsilon, b_2) y_3^k = \sum_{p,q \geq 0} u_{pq} \varepsilon^p y_3^q,$$

where $u_{00} = F_3(b_1(t), b_2(t))$ from (19)

Structure of Ω near the curve $\mathcal{J}$ of SP (13)

So $u_{00} = 0$,

$$u_{10} = \frac{\partial F\left(y_1, y_2\right)}{\partial y_1} = -\frac{512 y_2^4 \left(4y_2^2 - 3\right)^2 \left(4y_1 + 1\right)}{729} \stackrel{\text{def}}{=} H\left(y_1, y_2\right) \tag{20}$$

when $y_i = b_i\left(t\right)$, $i = 1{,}2$, and in general

$$u_{pq} = \frac{1}{p!} \cdot \frac{\partial^p T_q}{\partial y_1^p}, \text{ when } y_i = b_i\left(t\right), \ i = 1{,}2, \tag{21}$$

according to (17).

Structure of Ω near the curve $\mathcal{J}$ of SP (14)

Now, according to (20) and (18)

$$
\begin{aligned}
u_{10}(t) =& H\left(b_1(t), b_2(t)\right) = -\frac{\left(19t^2+22t+7\right)^3\left(13t^2+10t+1\right)^5}{497664\zeta^8},\\
u_{01}(t) =& G\left(b_1(t), b_2(t)\right) = \frac{\eta}{1728\zeta^4},\\
&\zeta = (t+1)(2t+1),\\
&\eta = 2224717t^8 + 12017960t^7 + 28029436t^6 + 37008760t^5 +\\
&\quad + 30350558t^4 + 15868120t^3 + 5174044t^2 + 963080t + 78397
\end{aligned}
$$

Structure of Ω near the curve $\mathcal{J}$ of SP (15)

By generalized Theorem 1 on the implicit function, the equation $u\left(\varepsilon, y_3\right) = 0$ has a solution as a power series over y_3

$$\varepsilon = \sum_{k=1}^{\infty} c_k\left(t\right) \cdot y_3^k, \tag{22}$$

where $c_k\left(t\right)$ are rational functions that are expressed through the coefficients $u_{pq}\left(t\right)$, which in turn are expressed through $b_1(t)$ and $b_2\left(t\right)$ according to (21). This expansion is valid for all values of t, except maybe the roots of the function $H(t)$. They correspond to points $y_1 = -1/4$, $y_2 = \pm\sqrt{3}/2$.

Structure of Ω near the curve $\mathcal{J}$ of SP (16)

Therefore, we have to remove them together with their neighborhoods. In particular,

$$c_1(t) = -\left(\frac{u_{01}}{u_{10}}\right) = -\frac{G}{H} = \left(288\eta\zeta^4\right) / \left(\left(19t^2 + 22t + 7\right) \times \right.$$
$$\times \left(6997t^6 + 24846t^5 + 37479t^4 + 30484t^3 + 13971t^2 + 3390t + 337\right) \times$$
$$\left. \times \left(13t^2 + 10t + 1\right)^4\right),$$

where the denominator has 2 real roots $t_{1,2} = \left(-5 \mp 2\sqrt{3}\right)/13$. According to (22) approximate $\varepsilon \approx c_1(t)\, y_3$.

Structure of Ω near the curve $\mathcal{J}$ of SP (17)

Let us return to the original coordinates, which for small $|y_3|$ on the variety Ω are approximated by

$$x_1 = (b_1(t) + c_1(t)\, y_3)\, y_3,\ x_2 = b_2(t)\,,\ x_3 = y_3, \qquad (23)$$

in which case

$$A_1 = x_1 - x_2 - \frac{1}{2},\ A_2 = x_1 + x_2 - \frac{1}{2},\ A_3 = \frac{1}{2} + y_3. \qquad (24)$$

Structure of Ω near the curve $\mathcal{J}$ of SP (18)

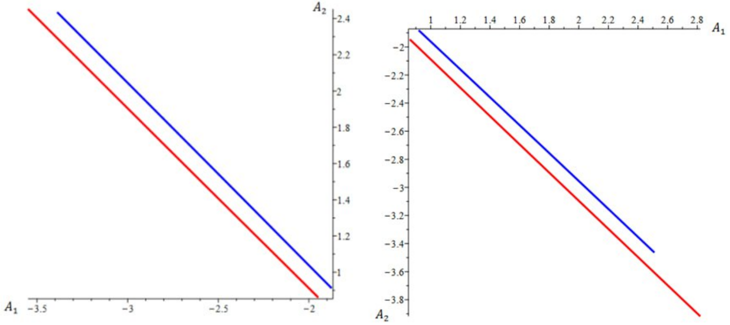

Figure 8: Curves (23), (24) at $y_3 = 1/20$.

Structure of Ω near the curve $\mathcal{J}$ of SP (19)

Fig. 8 at $y_3 = 1/20$ (i.e., $A_3 = 11/20$) shows the upper and lower sections of the curve (23), (24) for $1.4 < |b_2(t)| < 3$. The sections where $|b_2(t)| < 1.4$ are discarded, because they are affected by singularities of the SPs $P_4^{(3)}$ and $P_5^{(3)}$. We see that these curves are like parallel line segments and almost coincide. In the corresponding $A_3 = 0.505$ Fig. 12 in [Bruno, Batkhin, 2015] similar branches merge.

Fig. 9 shows the upper and lower sections of the curve (23), (24) at $y_3 = -1/20$ (i.e., $A_3 = 9/20$). Here the distance between the branches is larger, which corresponds to Fig. 8 in [Bruno, Batkhin, 2015], with $A_3 = 0.45$, where these branches do not merge.

Structure of Ω near the curve $\mathcal{J}$ of SP (20)

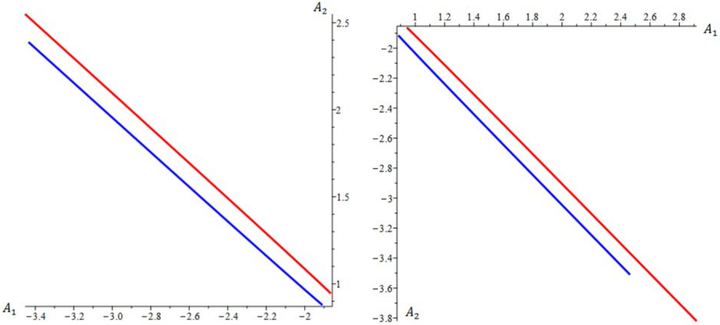

Figure 9: Curves (23), (24) at $y_3 = -1/20$.

References (1)

Bruno A. D. Power Geometry in Algebraic and Differential Equations. Amsterdam: Elsevier Science, 2000.

Bruno A. D. Algorithms for solving an algebraic equation. // Programming and Computer Software. 2018. Vol. 44, no. 6. P. 533–545. DOI: 10.1134/S0361768819100013.

Bruno A. D., *Azimov A. A.* Computation of unimodular matrices of power transformations. // Programming and Computer Software. 2023. Vol. 49, no. 1. P. 32–41. DOI: 10.1134/S0361768823010036. URL: https://disk.yandex.ru/i/zXlMlHu91CcQiw.

Bruno A. D., *Batkhin A. B.* Resolution of an algebraic singularity by power geometry algorithms. // Programming and Computer Software. 2012. Vol. 38, no. 2. P. 57–72. DOI: 10.1134/S036176881202003X. URL: https://disk.yandex.ru/i/dmylwZQZxQ_wTw.

References (2)

Bruno A. D., *Batkhin A. B.* Investigation of a real algebraic surface. // Programming and Computer Software. 2015. Vol. 41, no. 2. P. 74–82.

Nikonorov Y. G. Classification of generalized Wallach spaces. // Geometriae Dedicata. 2016. Vol. 111, no. 1. P. 193–212. DOI: 10.1007/s10711-015-0119-z.

Thompson I. Understanding Maple. Cambridge University Press, 2016. 228 p.

Thanks for your attention!