Parametric expansions of an algebraic variety near its singularities

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Abstract. Now there is a method, based on Power Geometry, that allows to find asymptotic forms and asymptotic expansions of solutions to different kinds of non-linear equations near their singularities. The method contains three algorithms: (1) Reducing equation to its normal form, (2) Separating truncated equations, (3) Power transformations of coordinates. Here we describe the method for the simplest case: a single algebraic equation, and apply it to an algebraic variety, described by an algebraic equation of order 12 in three variables. The variety was considered in study of Einstein's metrics and has several singular points and singular curves. Near some of them we compute a local parametric expansion of the variety.

1. Introduction

Here we propose a new method for solution of a polynomial equation

$$f(x_1, \dots, x_n) = 0$$

near its singular point. In the example we show computations of the method for a certain polynomial f and $n = 3$. The method is used:

I: The *Newton polyhedron* for separation of truncated equations and

II: *Power transformations* for simplification of these equations.

Here the basic ideas of this method are explained for the simplest case: a single algebraic equation. In Section 2 we give a generalization of Implicit Function Theorem. In Sections 3 and 4 we remind some constructions of Power Geometry [1]. In Section 5 we explain a way of computation of asymptotic parametric expansions of solutions. In Section 6 we show a variety Ω and some its singularities.

2. The implicit function theorem

Let $X = (x_1, \dots, x_n)$, $Q = (q_1, \dots, q_n)$, then $X^Q = x_1^{q_1}, \dots, x_n^{q_n}$, $\|Q\| = \sum_{j=1}^{n} q_j$.

Theorem 1. *Let*

$$f(X,\varepsilon,T)=\Sigma a_{Q,r}(T)X^Q\varepsilon^r,$$

where $0\le Q\in\mathbb{Z}^n$, $0\le r\in\mathbb{Z}$, *the sum is finite and* $a_{Q,r}(T)$ *are some functions of* $T=(t_1,\dots,t_m)$, *besides* $a_{00}(T)\equiv 0$, $a_{01}(T)\not\equiv 0$. *Then the solution to the equation* $f(X,\varepsilon,T)=0$ *has the form*

$$\varepsilon=\Sigma b_R(T)X^R, \tag{1}$$

where $0\le R\in\mathbb{Z}^n$, $0<\|R\|$, *the coefficients* $b_R(T)$ *are functions on* T *that are polynomials from* $a_{Q,r}(T)$ *with* $\|Q\|+r\le\|R\|$ *divided by* $a_{01}^{2\|R\|-1}$. *The expansion* (1) *is unique.*

This is a generalization of Theorem 1.1 of [1, Ch. II] on the implicit function and simultaneously a theorem on reducing the algebraic equation $f=0$ to its normal form (1) when the linear part $a_{01}(T)\not\equiv 0$ is non degenerate. In it, we must exclude the values of T near the zeros of the function $a_{01}(T)$.

Let $X=(x_1,\dots,x_n)\in\mathbb{R}^n$ or $\mathbb{C}^n$, and $f(X)$ be a polynomial. A point $X=X^0$, $f(X^0)=0$ is called *simple* if in it vector $(\partial f/\partial x_1,\dots,\partial f/\partial x_n)\neq 0$.

Definition 1. Let $\varphi(X)$ be some polynomial, $X=(x_1,\dots,x_n)$. A point $X=X^0$ of the set $\varphi(X)=0$ is called **singular point of the k-order**, if all partial derivatives of the polynomial $\varphi(X)$ for the $x_1,\dots,x_n$ turn into zero at this point, up to and including k-th order derivatives, and at least one partial derivative of order $k+1$ is nonzero.

3. The Newton polyhedron

Let the point $X^0=0$ be singular. Write the polynomial in the form $f(X)=\Sigma a_Q X^Q$, where $a_Q=\text{const}\in\mathbb{R}$, or $\mathbb{C}$. Let $\mathbf{S}(f)=\{Q: a_Q\neq 0\}\subset\mathbb{R}^n$.

The set $\mathbf{S}$ is called the *support* of the polynomial $f(X)$. Let it consist of points $Q_1,\dots,Q_k$. The convex hull of the support $\mathbf{S}(f)$ is the set

$$\Gamma(f)=\left\{Q=\sum_{j=1}^{k}\mu_jQ_j,\quad \mu_j\ge 0,\quad \sum_{j=1}^{k}\mu_j=1\right\},$$

which is called the *Newton polyhedron*.

Its boundary $\partial\Gamma(f)$ consists of generalized faces $\Gamma_j^{(d)}$, where d is its dimension of $0\le d\le n-1$ and j is its number. Numbering is unique for all dimensions d.

Each (generalized) face $\Gamma_j^{(d)}$ corresponds to its:

- *boundary subset* $\mathbf{S}_j^{(d)}=\mathbf{S}\cap\Gamma_j^{(d)}$,
- *truncated polynomial* $\hat f_j^{(d)}(X)=\Sigma a_QX^Q$ over $Q\in\mathbf{S}_j^{(d)}$, and
- *normal cone*
  $\mathbf{U}_j^{(d)}=\left\{P:\langle P,Q'\rangle=\langle P,Q''\rangle>\langle P,Q'''\rangle, Q',Q''\in\mathbf{S}_j^{(d)}, Q'''\in\mathbf{S}\backslash\mathbf{S}_j^{(d)}\right\}$,
  where $P=(p_1,\dots,p_n)\in\mathbb{R}_*^n$, the space $\mathbb{R}_*^n$ is conjugate (dual) to the space $\mathbb{R}^n$ and $\langle P,Q\rangle=p_1q_1+\ldots+p_nq_n$ is the scalar product.

At $X \to 0$ solutions to the full equation $f(X) = 0$ tend to non-trivial solutions of those truncated equations $\hat{f}_j^{(d)}(X) = 0$ whose normal cone $\mathbf{U}_j^{(d)}$ intersects with the negative orthant $P \leq 0$ in $\mathbb{R}_*^n$.

4. Power transformations

Let $\ln X = (\ln x_1, \dots, \ln x_n)$. The linear transformation of the logarithms of the coordinates

$$(\ln y_1, \dots, \ln y_n) \stackrel{\text{def}}{=} \ln Y = (\ln X)\alpha, \tag{2}$$

where α is a nondegenerate square n-matrix, is called *power transformation.*

By the power transformation (2), the monomial X^Q tranforms into the monomial Y^R, where $R = Q(\alpha^*)^{-1}$ and the asterisk indicates a transposition.

A matrix α is called *unimodular* if all its elements are integers and $\det \alpha = \pm 1$. For an unimodular matrix α, its inverse α^{-1} and transpose α^* are also unimodular.

Theorem 2. *For the face $\Gamma_j^{(d)}$ there exists a power transformation* (2) *with the unimodular matrix α which reduces the truncated sum $\hat{f}_j^{(d)}(X)$ to the sum from d coordinates, that is, $\hat{f}_j^{(d)}(X) = Y^S \hat{g}_j^{(d)}(Y)$ where $\hat{g}_j^{(d)}(Y) \equiv \hat{g}_j^{(d)}(y_1, \dots, y_d)$ is a polynomial. Here $S \in \mathbb{Z}^n$. The additional coordinates $y_{d+1}, \dots, y_n$ are local (small).*

The article [2] specifies an algorithm for computing the unimodular matrix α of Theorem 2.

5. Parametric expansion of solutions

Let $\Gamma_j^{(d)}$ be a face of the Newton polyhedron $\Gamma(f)$. Let the full equation $f(X) = 0$ is changed into the equation $g(Y) = 0$ after the power transformation of Theorem 2. Thus $\hat{g}_j^{(d)}(y_1, \dots, y_d) = g(y_1, \dots, y_d, 0, \dots, 0)$.

Let the polynomial $\hat{g}_j^{(d)}$ be the product of several irreducible polynomials

$$\hat{g}_j^{(d)} = \prod_{k=1}^{m} h_k^{l_k}(y_1, \dots, y_d), \tag{3}$$

where $0 < l_k \in \mathbb{Z}$. Let the polynomial h_k be one of them. Three cases are possible:
Case 1. The equation $h_k = 0$ has a polynomial solution $y_d = \varphi(y_1, \dots, y_{d-1})$. Then in the full polynomial $g(Y)$ let us substitute the coordinates $y_d = \varphi + z_d$, for the resulting polynomial $h(y_1, \dots, y_{d-1}, z_d, y_{d+1} \dots, y_n)$ again construct the Newton polyhedron, separate the truncated polynomials, etc. Such calculations were made in [3] and were shown in Introduction to [1].

Case 2. The equation $h_k = 0$ has no polynomial solution, but has a parametrization of solutions $y_j = \varphi_j(T), j = 1, \dots, d, \quad T = (t_1, \dots, t_{d-1})$.

Then in the full polynomial $g(Y)$ we substitute the coordinates

$$y_j = \varphi_i(T) + \beta_j \varepsilon, \quad j = 1, \ldots, d, \tag{4}$$

where $\beta_j = \text{const}$, $\Sigma\,|\beta_j| \neq 0$, and from the full polynomial $g(Y)$ we get the polynomial

$$h = \Sigma a_{Q'',r}(T) Y''^{Q''} \varepsilon^r, \tag{5}$$

where $Y'' = (y_{d+1}, \ldots, y_n)$, $0 \leq Q'' = (q_{d+1}, \ldots, q_n) \in \mathbb{Z}^{n-d}$, $0 \leq r \in \mathbb{Z}$. Thus $a_{00}(T) \equiv 0$, $a_{01}(T) = \sum_{j=1}^{d} \beta_j \partial \hat{g}_j^{(d)} / \partial y_j(T)$.

If in the expansion (3) $l_k = 1$, then $a_{01} \not\equiv 0$. By Theorem 1, all solutions to the equation $h = 0$ have the form $\varepsilon = \Sigma b_{Q''}(T) Y''^{Q''}$, i.e., according to (4) the solutions to the equation $g = 0$ have the form $y_j = \varphi_j(T) + \beta_j \Sigma b_{Q''}(T) Y''^{Q''}$, $j = 1, \ldots, d$. Such calculations were proposed in [4].

If in (3) $l_k > 1$, then in (5) $a_{01}(T) \equiv 0$ and for the polynomial (5) from Y'', ε we construct the Newton polyhedron by support $\mathbf{S}(h) = \{Q'', r : a_{Q'',r}(T) \not\equiv 0\}$, separate the truncations and so on.

Case 3. The equation $h_k = 0$ has neither a polynomial solution nor a parametric one. Then, using Hadamard's polyhedron [4], one can compute a piecewise approximate parametric solution to the equation $h_k = 0$ and look for an approximate parametric expansion.

Similarly, one can study the position of an algebraic manifold in infinity.

6. Variety Ω and its singularities

In [5], the investigation of the three-parametric family of special homogeneous spaces from the viewpoint of the normalized Ricci flow was started. The Ricci flows describe the evolution of Einstein's metrics on a variety. The equations of the normalized Ricci flow are reduced to a system of two differential equations with three parameters a_1, a_2 and a_3:

$$dx_j/dt = \tilde{f}_1(x_1, x_2, a_1, a_2, a_3), \quad j = 1, 2, \tag{6}$$

here $\tilde{f}_1$ and $\tilde{f}_2$ are certain functions. The singular point of this system are associated with invariant Einstein's metrics. At the singular (stationary) point x_1^0, x_2^0, system (6) has two eigenvalues λ_1 and λ_2. If at least one of them is equal to zero, then the singular (fixed) point x_1^0, x_2^0 is said to be *degenerate*. It was proved in [5] that the set Ω of the values of the parameters a_1, a_2, a_3 at which system (6) has at least one degenerate singular point is described by the equation

$$\begin{aligned}
Q(s_1, s_2, s_3) \equiv {} & (2s_1 + 4s_3 - 1)\big(64s_1^5 - 64s_1^4 + 8s_1^3 + 240s_1^2 s_3 - 1536 s_1 s_3^2 - \\
& - 4096 s_3^3 + 12 s_1^2 - 240 s_1 s_3 + 768 s_3^2 - 6 s_1 + 60 s_3 + 1\big) - 8 s_1 s_2 (2s_1 + 4s_3 - 1) \times \\
& \times (2s_1 - 32 s_3 - 1)(10 s_1 + 32 s_3 - 5) - 16 s_1^2 s_2^2 \big(52 s_1^2 + 640 s_1 s_3 + 1024 s_3^2 - 52 s_1 - \\
& - 320 s_3 + 13) + 64 (2s_1 - 1) s_2^3 (2s_1 - 32 s_3 - 1) + 2048 s_1 (2s_1 - 1) s_2^4 = 0,
\end{aligned}$$

where s_1, s_2, s_3 are elementary symmetric polynomials, equal respectively to $s_1 = a_1 + a_2 + a_3$, $s_2 = a_1a_2 + a_1a_3 + a_2a_3$, $s_3 = a_1a_2a_3$.

In [6], for symmetry reasons, the coordinates $\mathbf{a} = (a_1, a_2, a_3)$ were changed to the coordinates $\mathbf{A} = (A_1, A_2, A_3)$ by a linear transformation $\mathbf{a} = M\mathbf{A}$.

In [6] all singular points of the variety Ω in coordinates $\mathbf{A} = (A_1, A_2, A_3)$ were found. There are five points of the third order. Among them $P_1^{(3)} = (0, 0, 3/4)$. There are three second-order points and three algebraic curves of singular points of the first order. Among them is $\mathcal{I} = \{A_1 + A_2 + 1 = 0, A_3 = 1/2\}$.

In the talk we will consider the variety Ω in the neighborhood of point $P_1^{(3)}$ and curve $\mathcal{I}$. The methods proposed in [4] and described in Sections 2-5 are implemented.

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