On a property of Young diagrams of maximum dimensions

Egor Smirnov-Maltsev

People's Friendship University of Russia, Moscow, Russia

April 20, 2023

Dimension of a Young diagram

Dimension of a Young diagram A is the number of Young tableaux which A has. This work investigates the dependence of the dimension of a diagram A on difference between A and its base subdiagram.

5				
4	9			
3	8	12		
2	7	11	14	
1	6	10	13	15

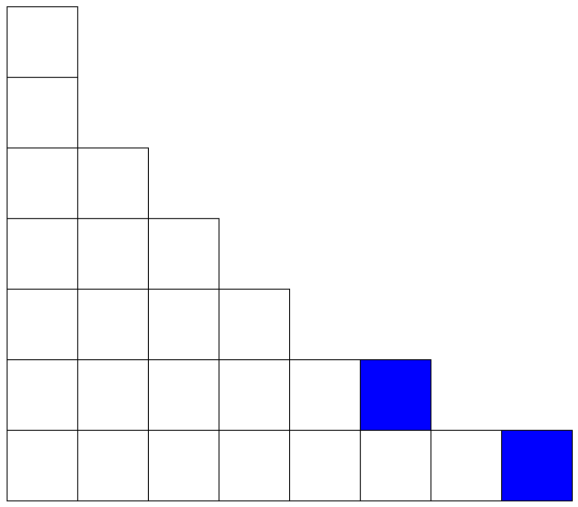
14				
13	15			
10	11	12		
6	7	8	9	
1	2	3	4	5

15				
10	14			
6	9	13		
3	5	8	12	
1	2	4	7	11

13				
10	15			
7	12	14		
4	6	9	11	
1	2	3	5	8

Dimension of a Young diagram

Base subdiagram of a diagram A is the maximum symmetric subdiagram of A.

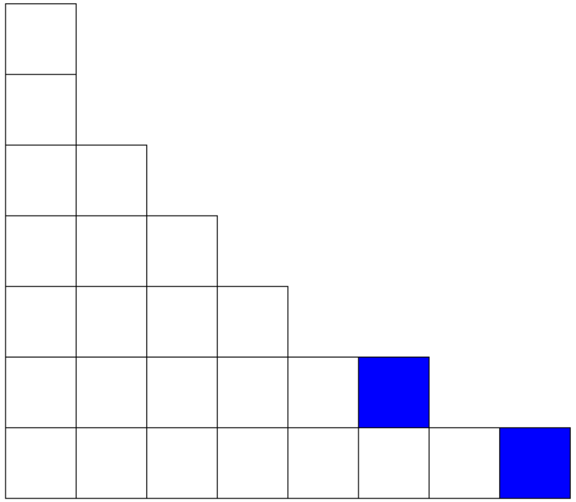


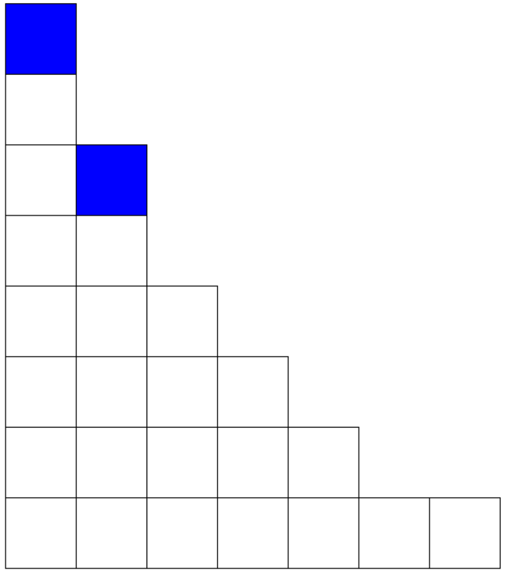
Motivation

The problem of finding a Young diagram of size n of maximum dimension was formulated in 1968 in Bayer and Brock's work and still remains unsolved. The dimension is a rational function of diagram shape. So we can reformulate the problem as searching for the maximum of this rational function.

Theorem on the maximum Young diagram

If a diagram A of size n is a diagram of maximum dimension, then it is the disjoint union of its base subdiagram A_{sym} and boxes located on one side of the line $y = x$. Moreover, if these boxes lie above $y = x$, then there is at most one box in each column. Similarly, if they lie below $y = x$, then there is at most one box in each row.





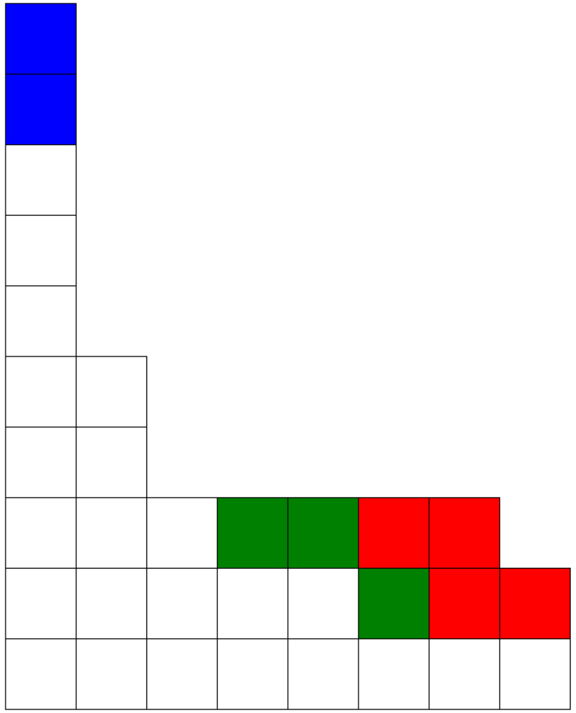
Proof of the theorem

Algorithm for modifying Young diagrams

An algorithm for modifying Young diagrams has been developed to prove the theorem. The idea of the algorithm is assumed that the algorithm modifies a diagram A of size n into another diagram B of size n so the dimension of B is greater than or equal to the dimension of A.

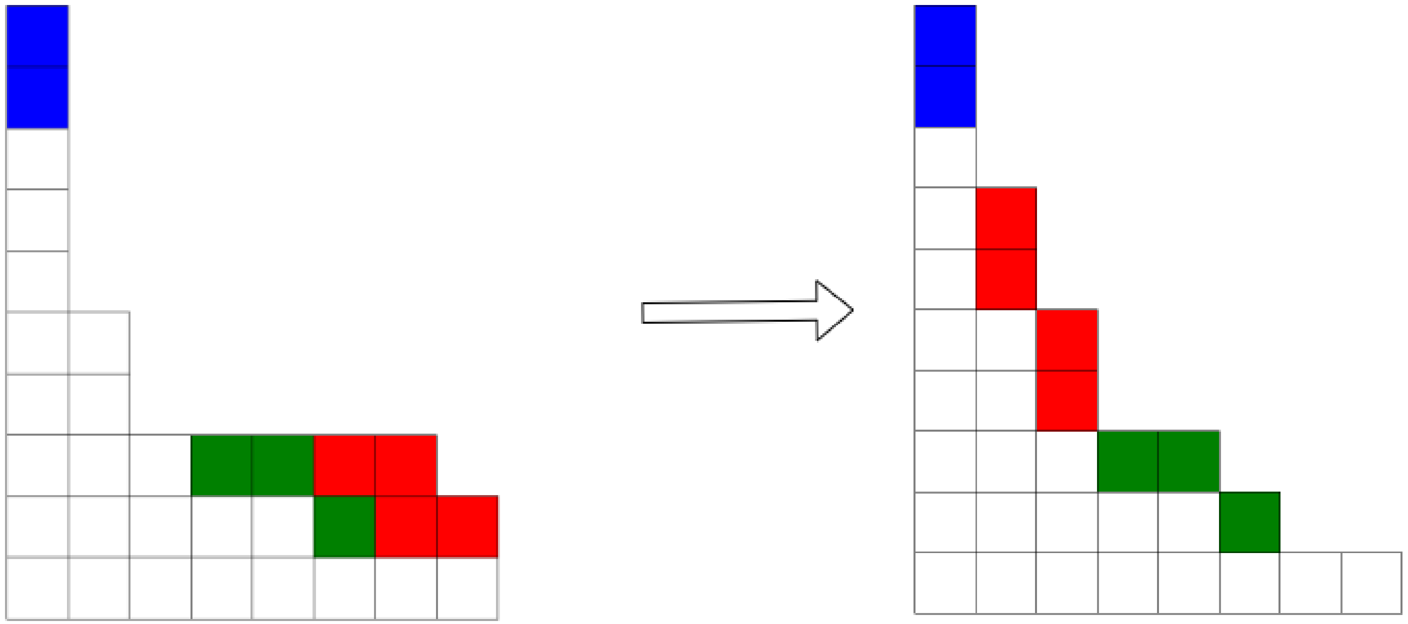
Algorithm for modifying Young diagrams

The algorithm receives a diagram A that consists of its base subdiagram A_{sym}, boxes $A_u \notin A_{sym}$ which lie above the line $y = x$ and boxes $A_d \notin A_{sym}$ which lie below $y = x$.



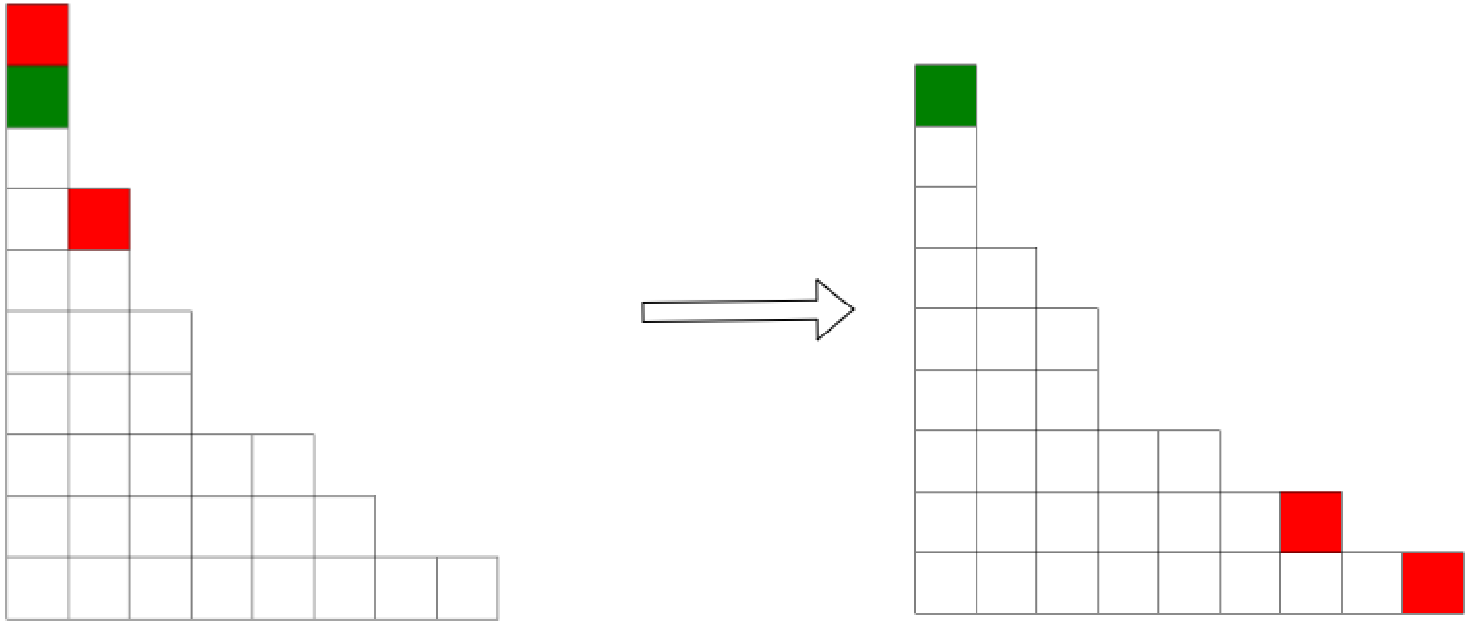
Algorithm for modifying Young diagrams

In the first step of the algorithm, we transform the diagram A into the diagram A_1 which has no boxes located below the line $y = x$ and not included in the base subdiagram A_1. In other words, A_1 consists only of its base subdiagram A_{1sym} and boxes $A_{1u} \notin A_{sym}$ located above $y = x$. Let us consider each row t containing boxes from A_d. If the t-th row has $2m$ boxes from A_d, we move m boxes from the t-th row to the t-th column and if the t-th row has $2m - 1$ boxes from A_d, we move m boxes to the t-th column.



Algorithm for modifying Young diagrams

In the second step, we transform the diagram A_1 into a diagram B that consists of its base subdiagram with single boxes added in some rows. All the added single boxes are located below the line $y = x$. The transformation algorithm is completely analogous to the transformation algorithm in the first step. Specifically, if a t-th column has $2m$ boxes from A_{1u}, we move m boxes from the t-th column to the t-th row and if the t-th column has $2m-1$ boxes from A_d, we move m boxes to the t-th row.



Dimensions of the original and modified diagrams

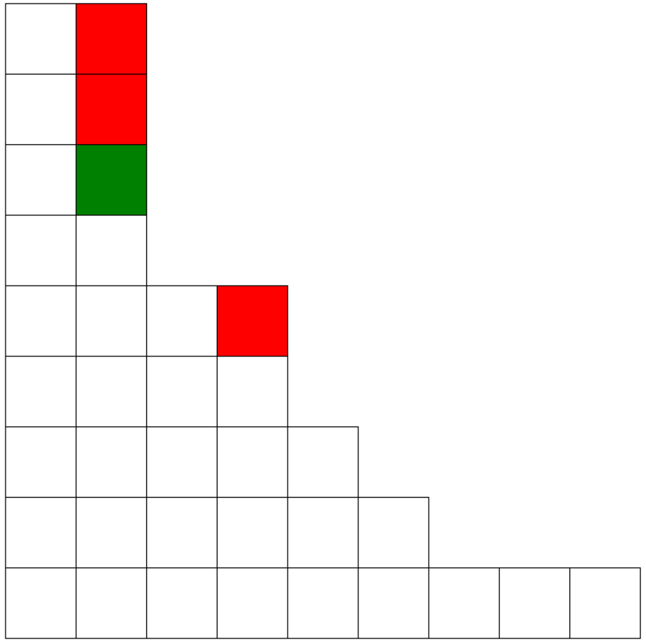
Here we prove that the dimension of a diagram does not decrease during each of the above transformations. This proof relies on the hook length formula:

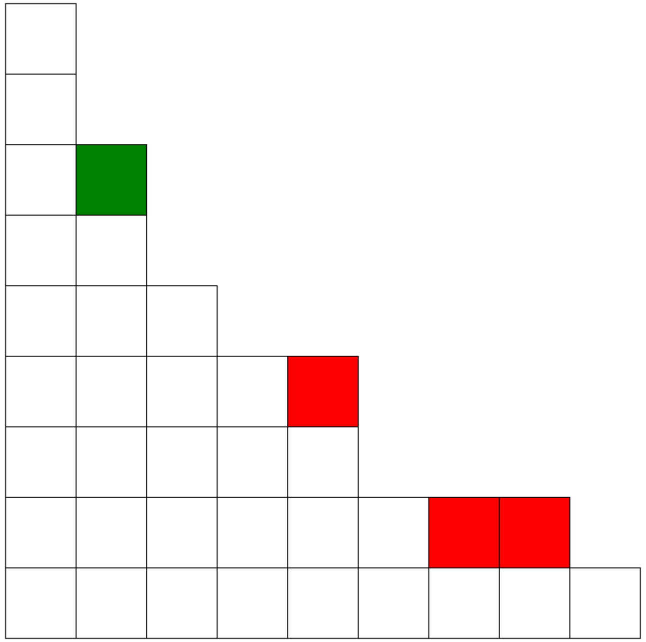
$$\dim(A) = \frac{n!}{\prod_{(i,j)\in A} h(i,j)}, \tag{1}$$

where A is a diagram of size n, $h(i,j)$ is the hook length of box (i,j) in diagram A.

Dimensions of the original and modified diagrams

Firstly, let us consider the second transformation.





Dimensions of the original and modified diagrams

The formula (1) implies that

$$\dim(A_1) \le \dim(B)$$

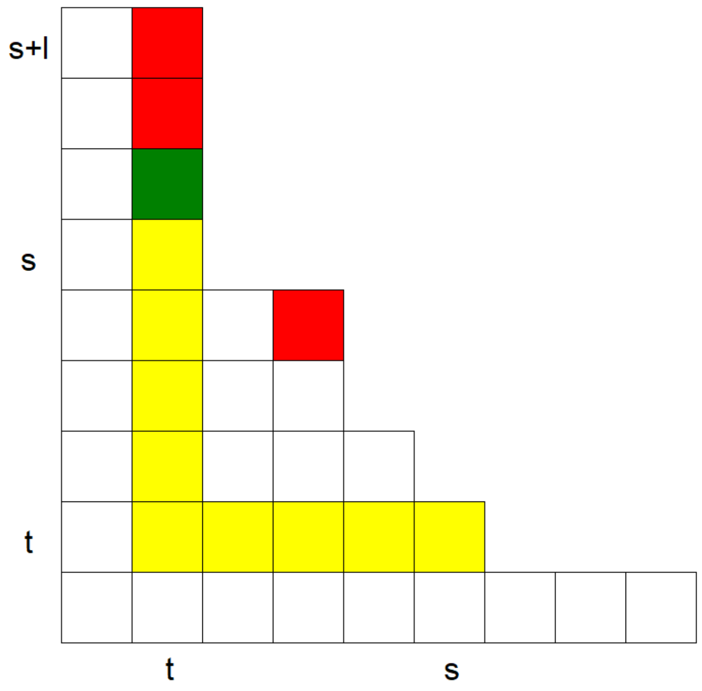
is equivalent to

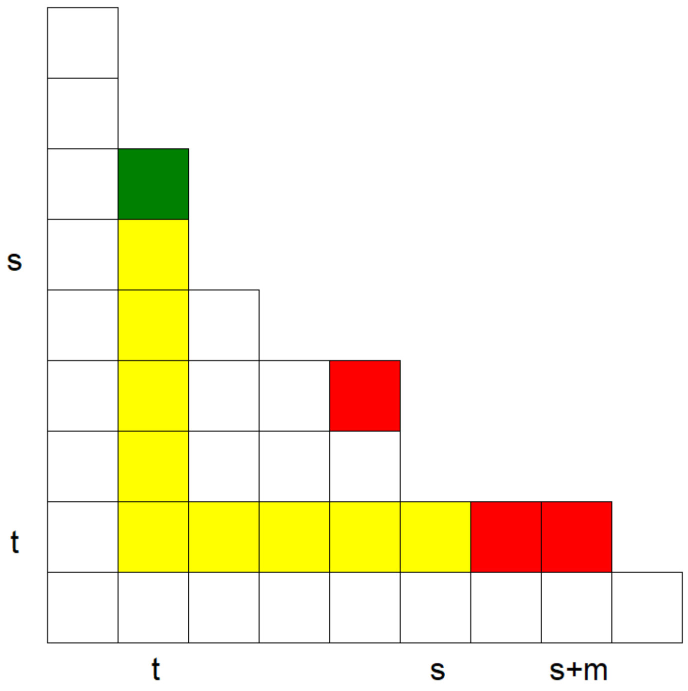
$$\frac{\prod\limits_{(i,j)\in A_1} h_1(i,j)}{\prod\limits_{(i,j)\in B} h_B(i,j)} \ge 1, \qquad (2)$$

where $h_1(i,j)$ and $h_B(i,j)$ are hook lengths of box (i,j) in diagram A_1 and B respectively.

Dimensions of the original and modified diagrams

Consider a hook of a box (t, t) for some t. Let there are s boxes in the t-th column of the base subdiagram of A_1, l boxes from A_{1u} in the t-th column of A_1, and m of this l boxes we move to the t-th row. We shall prove that the product of hook lengths for the boxes in this hook in the diagram A_1 is greater than or equal to the product of hook lengths for the boxes in this hook in the diagram B.





Dimensions of the original and modified diagrams

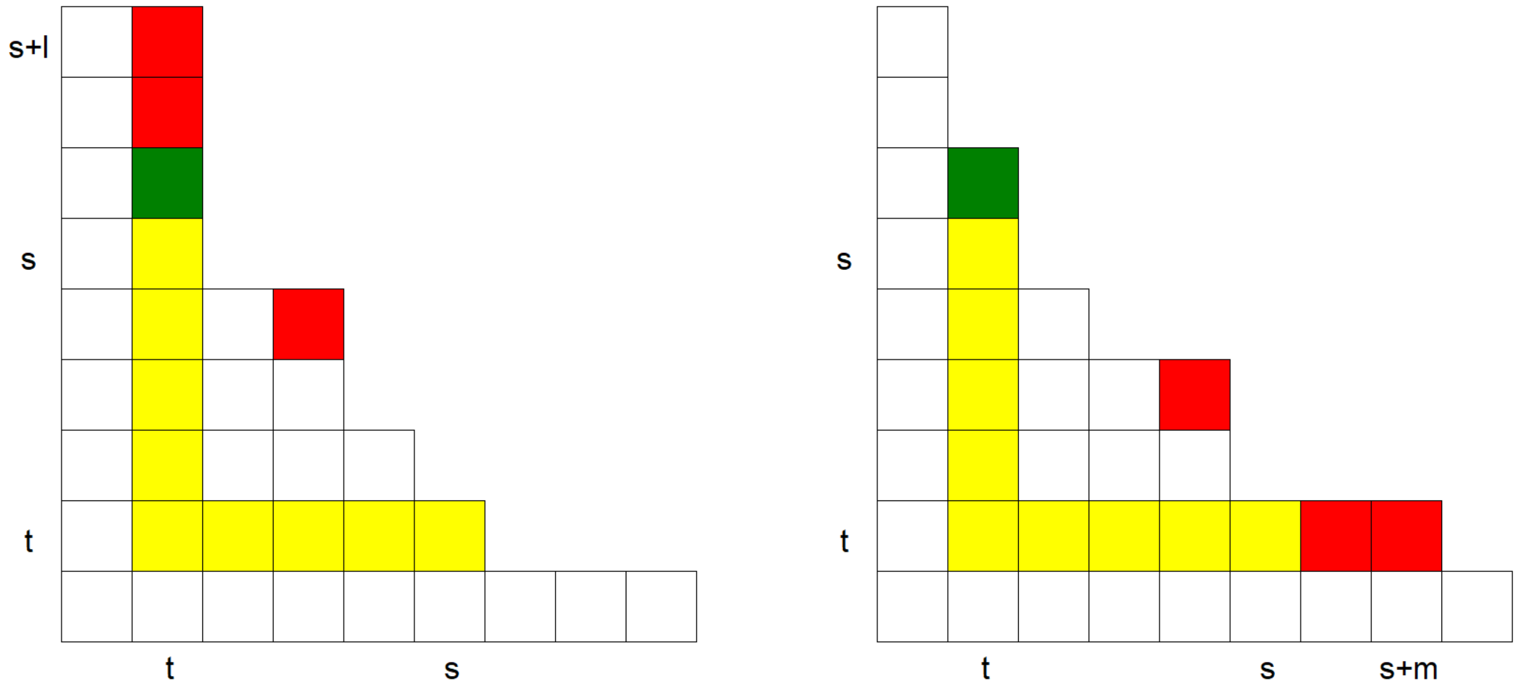
First, consider boxes contained in only one of these 2 diagrams. These are the boxes with coordinates $(t, s+l-m+1), (t, s+l-m+2), \ldots, (t, s+l)$ in the first diagram, as well as the boxes with coordinates $(s+1, t), (s+2, t), \ldots, (s+m, t)$ in the second diagram.

$$\frac{h_1(t, s+l-m+j)}{h_B(s+j, t)} \geq 1, \tag{3}$$

for each j from 1 to m.

Dimensions of the original and modified diagrams

Let us consider remaining boxes of the hook of (t,t) from A_{1u}. It is boxes with coordinates $(t, s+1), (t, s+2), \ldots, (t, s+l-m)$. $h_1(t, s+j) \geq h_B(t, s+j) + m$ for each j from 1 to m.



Dimensions of the original and modified diagrams

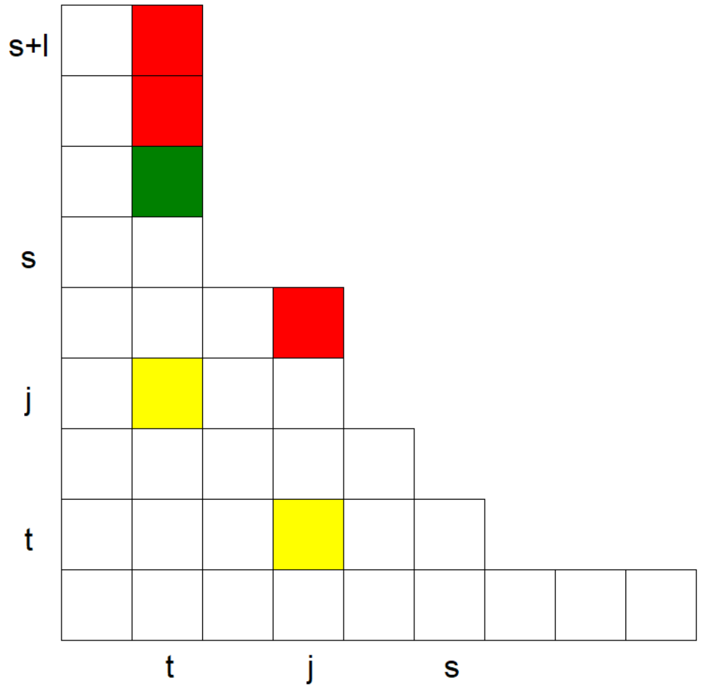
Let there are k boxes to the right of the box $(t, s+1)$ in the diagram B. Therefore

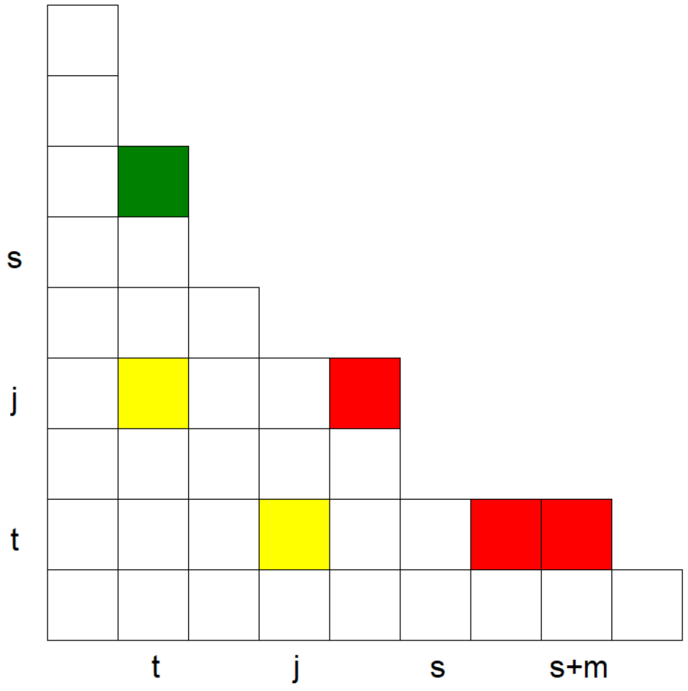
$$\frac{h_1(t,s+j)}{h_B(t,s+j)} \geq \frac{h_B(t,s+j)+m}{h_B(t,s+j)} \geq \frac{k+l-j+1}{k+l-j+1-m}$$

$$\frac{\prod_{j=1}^{l-m} h_1(s+j,t)}{\prod_{j=1}^{l-m} h_B(s+j,t)} \geq \frac{(l+k)! \cdot k!}{(k+m)! \cdot (k+l-m)!} \tag{4}$$

Dimensions of the original and modified diagrams

For $j \leq s$: $h_1(t,j) \geq h_B(j,t) + l - m$, $h_1(j,t) \geq h_B(t,j) - l + m$.





Dimensions of the original and modified diagrams

Let us prove that

$$\prod_{j=1}^{s} \frac{h_1(j,t) \cdot h_1(t,j)}{h_B(j,t) \cdot h_B(t,j)} \geq \prod_{j=1}^{s} \frac{(h_B(t,j) + l - m) \cdot (h_B(t,j) - l + m)}{h_B(t,j) \cdot h_B(t,j)} \qquad (5)$$

Let $h_B(j,t) = h_B(t,j) + r$ for some j. Then there are r boxes in j-th column of B that do not belong to B_{sym}.

Dimensions of the original and modified diagrams

Let the upper box of this r boxes has coordinates $(j, \tilde{t})$ and the upper box of the column $\tilde{t} > t$ of the diagram A_1 has coordinates $(\tilde{t}, \tilde{j})$. So $j \leq \tilde{j}$.

Dimensions of the original and modified diagrams

Let us delete the box $(j, \tilde{t})$ from the diagram B and the box $(\tilde{t}, \tilde{j})$.

$$\prod_{j=1}^{s} \frac{h_1(j,t) \cdot h_1(t,j)}{h_B(j,t) \cdot h_B(t,j)}$$

multiplied by

$$\frac{h_B(j,t)}{h_B(j,t)-1} \cdot \frac{h_B(t,\tilde{t})}{h_B(t,\tilde{t})-1} \cdot \frac{h_1(\tilde{t},t)-1}{h_1(\tilde{t},t)} \cdot \frac{h_1(t,\tilde{j})-1}{h_1(t,\tilde{j})} \leq 1$$

So if we delete all such boxes we obtain (5).

Dimensions of the original and modified diagrams

Box $(j, \tilde{t})$ in the diagram B corresponds to box $(\tilde{t}, \tilde{j})$ in the diagram A_1 where $\tilde{j} \geq j$. So

$$\frac{h_1(j,t)}{h_B(j,t)} \cdot \frac{h_B(t,j)}{(h_B(t,j)+l-m)} \geq \frac{(h_B(t,j)+r+l-m)\cdot h_B(t,j)}{(h_B(t,j)+r)\cdot(h_B(t,j)+l-m)} =$$

$$= \prod_{i=1}^{r} \frac{(h_B(t,j)+i+l-m)\cdot(h_B(t,j)+i)}{(h_B(t,j)+i-1+l-m)\cdot(h_B(t,j)+i-1)}$$

Dimensions of the original and modified diagrams

$$\prod_{j=1}^{s} \frac{h_1(j,t)\cdot h_1(t,j)}{h_B(j,t)\cdot h_B(t,j)} \geq \prod_{j=1}^{s} \frac{(h_B(t,j)+l-m)\cdot(h_B(t,j)-l+m)}{h_B(t,j)\cdot h_B(t,j)} \geq$$

$$\geq \prod_{x=k+m+1}^{n} \frac{(x+l-m)\cdot(x-l+m)}{x\cdot x} = \frac{(n+l-m)!\cdot(n-l+m)!\cdot(k+m)!\cdot(k+m)!}{(k+l)!\cdot(k+2m-l)!\cdot n!\cdot n!} \geq$$

$$\geq \frac{(k+m)!\cdot(k+m)!}{(k+l)!\cdot(k+2m-l)!} \tag{6}$$

Dimensions of the original and modified diagrams

Multiplying inequalities 3, 4 and 6 yields

$$\frac{\prod\limits_{(i,j)\in h_1(t,t)} h_1(i,j)}{\prod\limits_{(i,j)\in h_B(t,t)} h_B(i,j)} \geq \frac{(k+m)!}{(k+2m-l)!} \cdot \frac{k!}{(k+l-m)!}$$

If $2m = l$ then

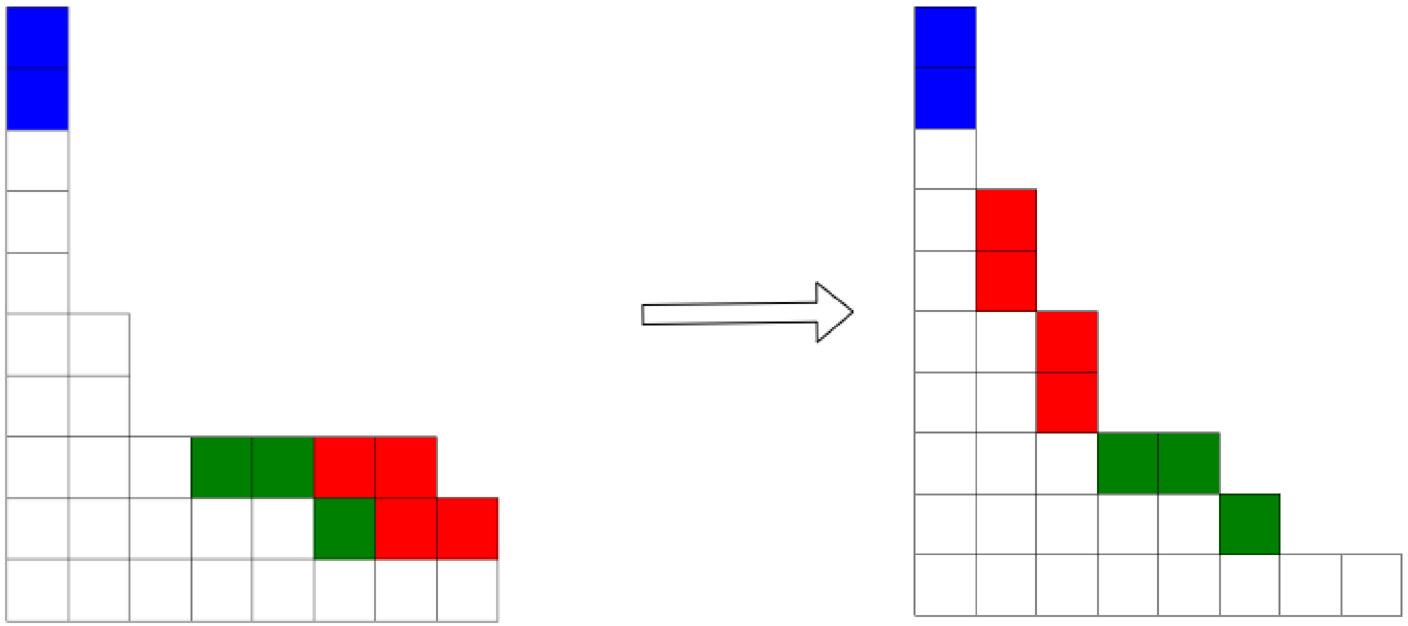
$$\frac{(k+m)!}{(k+2m-l)!} \cdot \frac{k!}{(k+l-m)!} = 1$$

If $2m = l + 1$ then

$$\frac{(k+m)!}{(k+2m-l)!} \cdot \frac{k!}{(k+l-m)!} = \frac{k+m}{k+1} \geq 1$$

Dimensions of the original and modified diagrams

The proof that the dimension of the diagram does not decrease during the first transformation comes from the previous statements. Particularly, it can be claimed that $\dim(A \setminus A_u) \le \dim(A_1 \setminus A_u)$. Then, we add boxes one by one from A_u to both diagrams. The hook lengths product of boxes of A grows faster than the hook lengths product of boxes of A_1 each time a box is added.



Conclusion

We can significantly improve algorithms for finding diagrams of maximum dimension using the theorem. Moreover, the algorithm for modifying Young diagrams enables to construct diagrams with large dimension.

Thanks for your attention