On a property of Young diagrams of maximum dimensions

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Abstract. The work is devoted to finding the Young diagrams of large dimensions, i.e. those which have large number of Young tableaux. The algorithm which modifies a diagram A_n of size n into another diagram A_{1n} of size n is proposed. It is proved that the dimension of A_{1n} is greater than or equal to the dimension of A_n. A criterion for a Young diagram of maximum dimension is formulated.

Introduction

This work is devoted to investigation of an open combinatorial problem [1] of finding a Young diagram of size n with the maximum dimension. In other words, the goal is to find a Young diagram which has the largest number of Young tableaux among all the diagrams of size n. The dimension is a rational function of diagram shape. So we can reformulate the problem as search for the maximum of this rational function. This talk is dedicated to an important property of a diagram of maximum dimension.

Let us define a basic subdiagram of a diagram A as the maximum symmetric subdiagram of A. So each diagram A consists of its basic subdiagram A_{sym}, boxes A_d located below the line $y = x$ and not included in the basic subdiagram, as well as boxes A_u located above $y = x$ and not included in the basic subdiagram. The study of Young diagrams of large dimension [2] has shown that a diagram A with the largest dimension has either no A_u or no A_d boxes. The idea of the algorithm proposed in this work has the similar nature with the previous algorithms [3]. It is assumed that the algorithm modifies a diagram A_n of size n into another diagram A_{1n} of size n so the dimension of A_{1n} is greater than or equal to the dimension of A_n.

The last statement can be proved using the hook length formula [4]:

$$\dim(A_n) = \frac{n!}{\prod\limits_{(i,j)\in A_n} h(i,j)}, \tag{1}$$

where A_n is a diagram of size n, $h(i,j)$ is the hook length of box (i,j) in diagram A_n. Our goal is to prove that $\dim(A_n) \leq \dim(A_{1n})$. Since we consider diagrams of the same size, it is enough to prove that

$$\prod_{(i,j)\in A_n} h(i,j) \geq \prod_{(i,j)\in A_{1n}} h_1(i,j), \tag{2}$$

where $h(i,j)$ is the hook length of a box (i,j) in diagram A_n, and $h_1(i,j)$ is the hook length of a box (i,j) in diagram A_{1n}.

1. Algorithm for modifying Young diagrams

In the first step of the algorithm, we transform a diagram A into the diagram A_1 which has no boxes located below the line $y = x$ and not included in the basic subdiagram A_1. In other words, A_1 consists only of its basic subdiagram A_{1sym} and boxes $A_{1u} \notin A_{sym}$ located above the $y = x$. Let us consider each row t containing boxes from A_d. We move half of such boxes in a row t to a column t. If the row t has $2l + 1$ boxes from A_d, we move $l + 1$ boxes. An example of this procedure is shown in Figure 1. The white boxes form the basic subdiagram of the diagram A, the moved boxes are highlighted in gray, and the remaining boxes are highlighted in black.

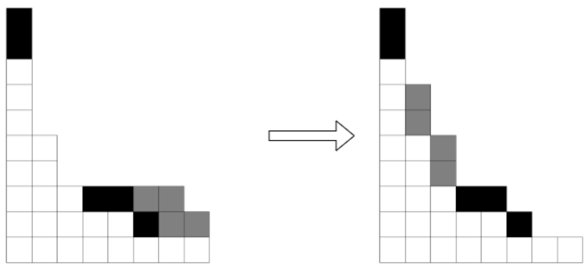

FIGURE 1. The first step of the algorithm

On the second step, we transform the diagram A_1 into a diagram A_2 that consists of its base subdiagram with single boxes added in some rows. All the added single boxes are located below the line $y = x$. An example of this step is illustrated in Figure 2. The meaning of the colors is the same as in the previous example.

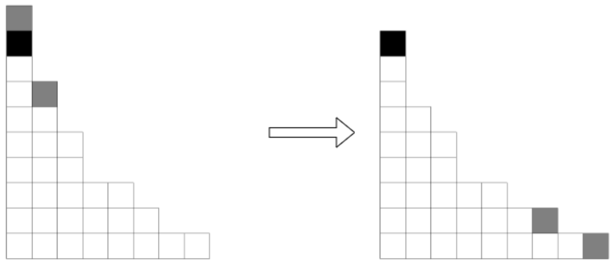

FIGURE 2. The second step of the algorithm

2. Dimensions of the original and modified diagrams

Here we prove that the dimension of a diagram does not decrease during each of the above transformations. Firstly, let us consider the second transformation. By formula 2, we need to prove that

$$\prod_{(i,j)\in A_1} h_1(i,j) \geq \prod_{(i,j)\in A_2} h_2(i,j). \tag{3}$$

Let t_i be all the columns that contain boxes from A_{1u}. Consider a column t_i for some i. t_i' is the row that is symmetric to the column t_i with respect to $y = x$. Then we can directly prove that the product of hook lengths for the boxes in t_i column and t_i' row in a diagram A_1 is not less than the product of hook lengths for the boxes in t_i column and t_i' row in a diagram A_2. Hook lengths for some boxes are counted several times but it does not affect the proof because the hook lengths of these boxes in the diagram A_1 are not greater than the hook lengths of these boxes in the diagram A_2. It is proved similarly that the product of hook lengths in the diagram A_1 for remaining boxes is greater than the product of hook lengths for these boxes in the diagram A_2.

The proof that the dimension of the diagram does not decrease during the first transformation comes from the previous statements. Particularly, it can be claimed that $\dim(A \setminus A_u) \leq \dim(A_1 \setminus A_u)$. Then, we add boxes one by one from A_u to both diagrams. The hook lengths product of boxes of A_1 grows faster than the hook lengths product of boxes of A_2 each time a box is added.

References

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