Approximation of the zeros of the Riemann zeta function by rational functions

Yuri Matiyasevich
Petersburg Department of Steklov Institute of Mathematics
St. Petersburg, 191023, Russia

https://logic.pdmi.ras.ru/~yumat/

Plan of the talk

1. Number-theoretical prerequisites

2. Traditional methods for calculating zeros

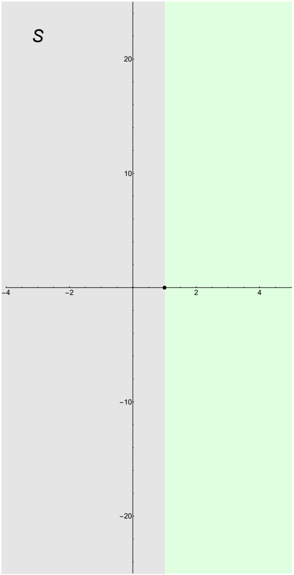
3. The problem

4. Our first solution

5. Approximations by Dirichlet polynomials

Part 1. Number-theoretical prerequisites

The Riemann zeta function



The Riemann zeta function can be defined by a *Dirichlet series*, namely,

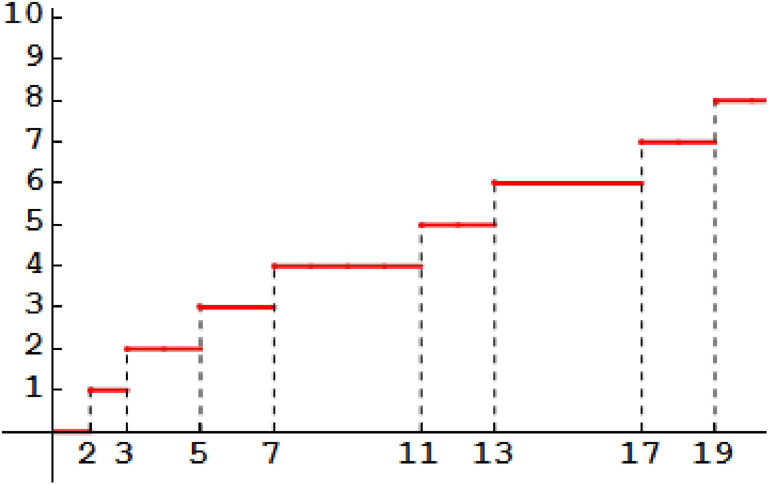
$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

The series converges for $\mathrm{Re}(s) > 1$ (the green half-plane) but the zeta function can be analytically extended to the whole complex plane with the exception of the simple pole at $s = 1$

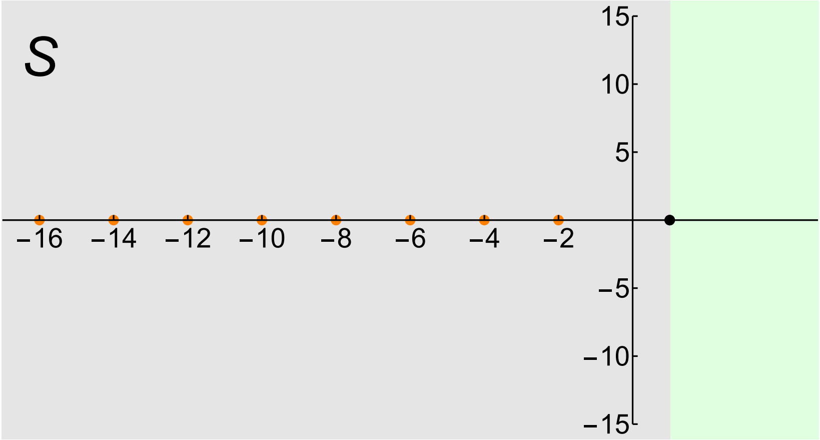
$$\zeta(1) = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots = \infty$$

How many primes are less or equal to a given bound x?

$$\pi(x) = \text{the number of primes } p \text{ such that } p \leq x$$



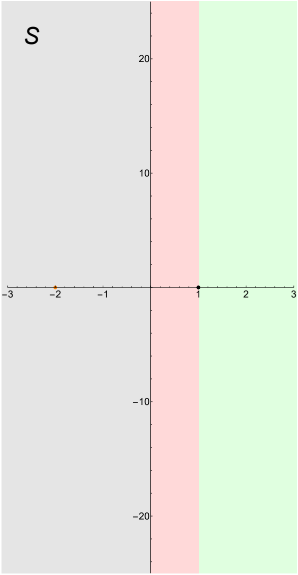
Trivial zeros of the zeta function



Euler: $0 = \zeta(-2) = \zeta(-4) = \cdots = \zeta(-2m) = \ldots$

Negative even integer called the *trivial zeros* of the zeta function

Non-trivial zeros of the zeta function



Critical strip (in pink)

Riemann (1859): *All other (called non-trivial zeros) of the zeta function are non-real and satisfy*

$$0 \leq \mathrm{Re}(s) \leq 1$$

J. Hadamard and Ch. de la Vallée Poussin (1896, idependently): *All the non-trivial zeta zeros satisfy*

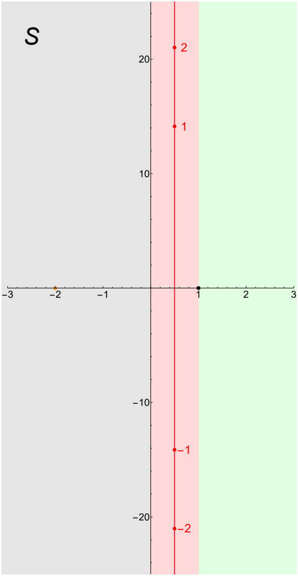
$$0 < \mathrm{Re}(s) < 1$$

Corollary (Prime Number Theorem)

$$\pi(x) = \mathrm{Li}(x) + o(x/\ln(x))$$

$$\mathrm{Li}(x) = \int_2^x \frac{\mathrm{d}y}{\ln(y)} \approx x/\ln(x)$$

The Riemann Hypothesis



Critical line $\mathrm{Re}(s) = \frac{1}{2}$

$$\rho_2 = 0.5 + 21.022039638771554992628...\mathrm{i}$$

$$\rho_1 = 0.5 + 14.134725141734693790457...\mathrm{i}$$

RH: *All the non-trivial zeta zeros satisfy*

$$\mathrm{Re}(s) = \frac{1}{2}$$

Equivalent formulation:

$$\pi(x) = \mathrm{Li}(x) + O(x^{1/2}\ln(x))$$

Part 2. Traditional methods for calculating zeros

Newton's iterations

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad \zeta(\rho) = 0 \qquad a \approx \rho$$

$$\begin{aligned} a_0 &= a \\ a_{n+1} &= a_n - \frac{\zeta(a_n)}{\zeta'(a_n)} \end{aligned}$$

$$a_n \underset{n\to\infty}{\longrightarrow} \rho$$

Drawback: difficult to analyze the limiting value of the process.

Taylor expansion

$$P_{a,N}(s) = \sum_{n=0}^{N} \frac{d_{a,n}}{n!}(s-a)^n \qquad d_{a,n} = \frac{\mathrm{d}^n}{\mathrm{d}s^n}\zeta(s)\Big|_{s=a}$$

$$\zeta(s) = P_{a,N}(s) + O\big((s-a)^{N+1}\big)$$

$$P_{a,N}(s) \approx \zeta(s) \qquad P_{a,N}(\rho) \approx \zeta(\rho) = 0$$

$$\zeta(s) = 0 \qquad\qquad P_{a,N}(s) = 0$$

Drawbacks.

1) No "explicit" expression for the roots of the equation (unless $N \leq 4$).

2) Which of the N roots of the equation is the desired approximation to ρ?

3) ρ should be closer to a than the pole of the zeta function at $s = 1$.

Part 3. The problem

The problem

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

$$a \approx \rho \quad \zeta(\rho) = 0$$

$$d_{a,n} = \frac{\mathrm{d}^n}{\mathrm{d}s^n}\zeta(s)\bigg|_{s=a}, \qquad n = 0, 1, \dots, N$$

Problem. *How numbers*

$$a, d_0, \dots, d_N$$

can be used for calculating a better approximation to the zeta zero ρ*?*

Part 4. Our first solution

My wild idea

$$\zeta(s) = 0 \qquad \zeta(s) = P_{a,N}(s) + O\big((s-a)^{N+1}\big) \qquad P_{a,N}(s) = 0$$

$$P_{a,N}(s) = \sum_{n=0}^{N} p_{a,N,n} s^n$$

Let us replace $s^1,\ s^2,\ \dots,\ s^N$ by independent unknowns $s_1,\ s_2,\ \dots,\ s_N$:

$$P_{a,N}(s_1,\dots,s_N) = p_{a,N,0} + \sum_{n=1}^{N} p_{a,N,n} s_n$$

$$\zeta(s) = 0 \qquad P_{a,N}(s) = 0 \qquad P_{a,N}(s_1,\dots,s_N) = 0$$

We *wish* that

$$s_2 = s_1^2,\ s_3 = s_1^3,\ \dots,\ s_N = s_1^N$$

or at least

$$s_2 \approx s_1^2,\ s_3 \approx s_1^3,\ \dots,\ s_N \approx s_1^N$$

Where could we get more equations?

$$m^{-s}\zeta(s) = 0$$

$$m^{-s}\zeta(s) = P_{a,N,m}(s) + O\big((s-a)^{N+1}\big)$$

$$m^{-s}\zeta(s) = 0 \qquad\qquad P_{a,N,m}(s) = 0$$

$$P_{a,N,m}(s) = p_{a,N,0,m} + \sum_{n=1}^{N} p_{a,N,n,m} s^n$$

$$P_{a,N,m}(s_1,\ldots,s_N) = p_{a,N,0,m} + \sum_{n=1}^{N} p_{a,N,n,m} s_n$$

$$m^{-s}\zeta(s) = 0 \qquad\qquad P_{a,N,m}(s_1,\ldots,s_N) = 0$$

Linear systems

$$m^{-s}\zeta(s) = 0, \qquad m = 1, \ldots N$$

$$P_{a,N,m}(s_1, \ldots, s_N) = 0, \qquad m = 1, \ldots N \tag{$*$}$$

Crucial questions:

- *Would the solution of system* $(*)$ *(if it exists and is unique) satisfy*

$$s_2 \approx s_1^2,\ s_3 \approx s_1^3,\ \ldots,\ s_N \approx s_1^N\ ?$$

- *How* s_1 *from this solution would be related to the zeros of the zeta function?*

s_1 is a rational function of a and $d_{a,n} = \frac{\mathrm{d}^n}{\mathrm{d}s^n}\zeta(s)\big|_{s=a}$, $n = 0, 1, \ldots, N$

Numerical example 1

$$\zeta(\rho_1) = 0 \qquad \rho_1 = 0.5 + 14.134725141734\ldots\mathrm{i}$$

$$a = 0.4 + 14\mathrm{i} \qquad N = 5$$

$$s_1 = 0.499999828490\ldots + 14.134725265432\ldots\mathrm{i}$$

$$\rho_1/s_1 = 1 + (-8.311... - 1.242...\mathrm{i}) \times 10^{-7}$$

$$\begin{aligned}
s_2/s_1^2 &= 1 + (6.228... - 4.386...\mathrm{i}) \times 10^{-9} \\
s_3/s_1^3 &= 1 + (1.733... - 1.486...\mathrm{i}) \times 10^{-8} \\
s_4/s_1^4 &= 1 + (3.171... - 3.286...\mathrm{i}) \times 10^{-8} \\
s_5/s_1^5 &= 1 + (4.749... - 5.954...\mathrm{i}) \times 10^{-8}
\end{aligned}$$

More data for $a = 0.4 + 14\mathrm{i}$

$$N = 50: \qquad \rho_1/s_1 = 1 + (-2.549641... + 4.473122...\mathrm{i}) \times 10^{-84}$$

$$N = 75: \qquad \rho_1/s_1 = 1 + (-0.789141... - 2.357178...) \times 10^{-124}$$

$$N = 100: \qquad \rho_1/s_1 = 1 + (-6.635292... - 0.802832...) \times 10^{-165}$$

$$N = 150: \qquad \rho_1/s_1 = 1 + (\ 5.089922... - 0.495039...) \times 10^{-246}$$

Numerical example 2

$$\zeta(\rho_2) = 0 \qquad \rho_2 = 0.5 + 21.0220396387\ldots\mathrm{i}$$

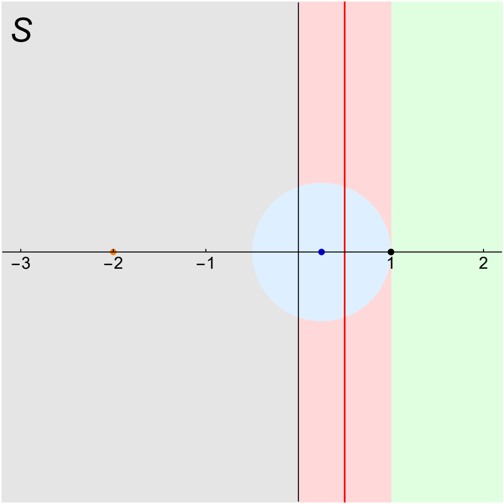
$$a = 0.4 + 21\mathrm{i} \qquad N = 5$$

$$s_1 = 0.499999989006\ldots + \ 21.022039625249\ldots\mathrm{i}$$

$$\rho_2/s_1 = 1 + (6.553... - 5.073...\mathrm{i}) \times 10^{-10}$$

$$\begin{aligned}
s_2/s_1^2 &= 1 + (1.301... + 2.431...\mathrm{i}) \times 10^{-10} \\
s_3/s_1^3 &= 1 + (4.351... + 7.115...\mathrm{i}) \times 10^{-10} \\
s_4/s_1^4 &= 1 + (.958... + 1.382...\mathrm{i}) \times 10^{-9} \\
s_5/s_1^5 &= 1 + (1.741... + 2.228...\mathrm{i}) \times 10^{-9}
\end{aligned}$$

Numerical example 3



$a = 0.25$ $\qquad N = 40$

$\zeta(-2) = 0$

$s_1 = \quad -2 \quad +1.2949... \times 10^{-13}$

$s_2 = \quad 4 \quad -7.8827... \times 10^{-13}$

$s_3 = \quad -8 \quad +3.4129... \times 10^{-12}$

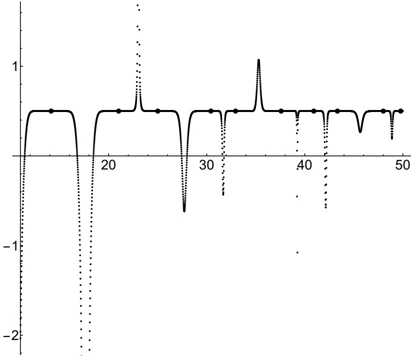
$s_4 = \quad 16 \quad -1.9992... \times 10^{-11}$

$s_5 = \quad -32 \quad -1.5676... \times 10^{-11}$

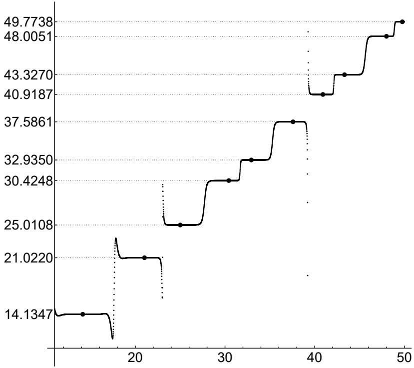
Away from zeros

$N = 10 \qquad a = 0.4 + \mathrm{i}\tau \qquad \tau$ from 10 to 50 with step 0.01

$\mathrm{Re}(s_1)$

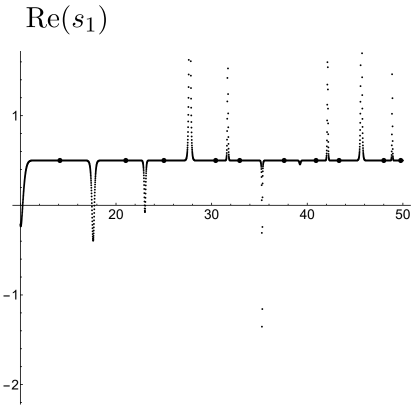


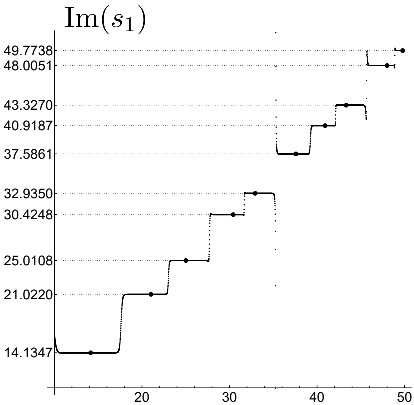
$\mathrm{Im}(s_1)$



Away from zeros

$N = 20 \qquad a = 0.4 + \mathrm{i}\tau \qquad \tau$ from 10 to 50 with step 0.01





Conjecture A. *For all a except for a set of zero measure there is a zero ρ of the zeta function such that for all k*

$$s_k \to \rho^k \text{ as } N \to \infty.$$

Part 4. Approximations by Dirichlet polynomials

Approximations by Dirichlet polynomials

$$m^{-s}\zeta(s) = P_{a,N,m}(s) + O\big((s-a)^{N+1}\big)$$

$$P_{a,N,m}(s) = p_{a,N,0,m} + \sum_{n=1}^{N} p_{a,N,n,m} s^n$$

$$P_{a,N,m}(s_1,\ldots,s_N) = p_{a,N,0,m} + \sum_{n=1}^{N} p_{a,N,n,m} s_n$$

$$P_{a,N,m}(s_1,\ldots,s_N) = 0, \quad m = 1,\ldots N$$

$$m^{-s}\zeta(s) = D_{a,N,m}(s) + O\big((s-a)^{N}\big)$$

$$D_{a,N,m}(s) = d_{a,N,1,m} + \sum_{n=2}^{N} d_{a,N,n,m} n^{-s}$$

$$D_{a,N,m}(s_2,\ldots,s_N) = d_{a,N,1,m} + \sum_{n=2}^{N} d_{a,N,n,m} n^{-s_n}$$

$$D_{a,N,m}(s_2,\ldots,s_N) = 0, \quad m = 1,\ldots N-1$$

Approximations by Dirichlet polynomials

$$D_{a,N,m}(s_2,\dots,s_N) = d_{a,N,1,m} + \sum_{n=2}^{N} d_{a,N,n,m} n^{-s_n}$$

$$D_{a,N,m}(s_2,\dots,s_N) = 0, \qquad m = 1,\dots N-1$$

n^{-s_n} is replaced by q_n:

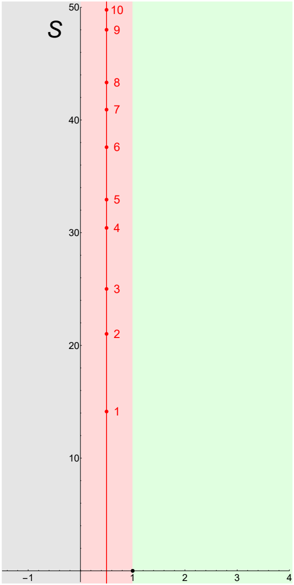
$$Q_{a,N,m}(q_2,\dots,q_N) = d_{a,N,1,m} + \sum_{n=2}^{N} d_{a,N,n,m} q_n$$

$$Q_{a,N,m}(q_2,\dots,q_N) = 0, \qquad m = 1,\dots N-1 \tag{$*$}$$

Crucial question: *Would the solution of system* $(*)$ *(if it exists and is unique) satisfy*

$$q_2 \approx 2^{-\rho},\ q_3 \approx 3^{-\rho},\ \dots,\ q_N \approx N^{-\rho}\ ?$$

Dirichlet images of the zeta zeros

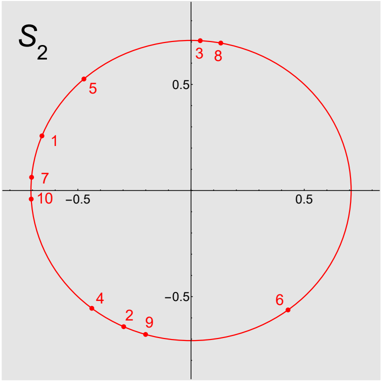


$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

n^{-s} is called the n*th Dirichlet image of* s

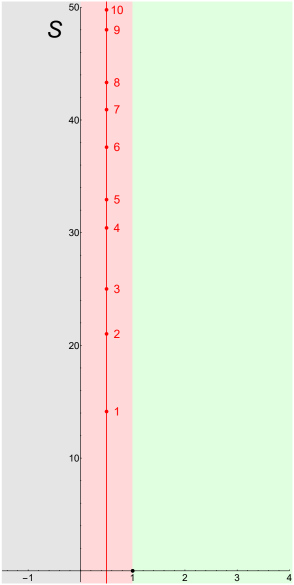
$s \mapsto 2^{-s}$

$|s| = 2^{-1/2}$



2nd Dirichlet images of the first 10 zeta zeros

Dirichlet images of the zeta zeros

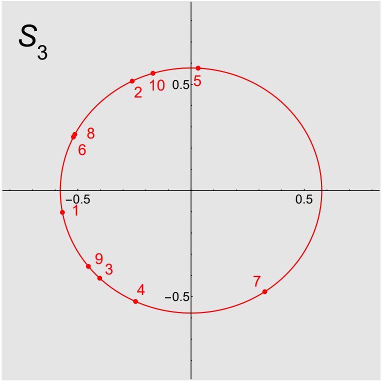


$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

n^{-s} is called the n*th Dirichlet image of* s

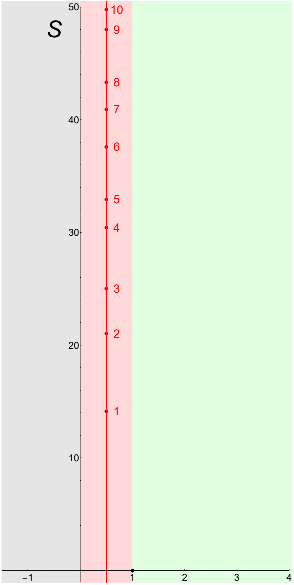
$$s \mapsto 3^{-s}$$

$$|s| = 3^{-1/2}$$



3rd Dirichlet images of the first 10 zeta zeros

Dirichlet images of the zeta zeros

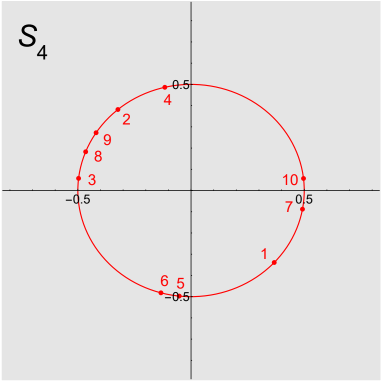


$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

n^{-s} is called the n*th Dirichlet image of* s

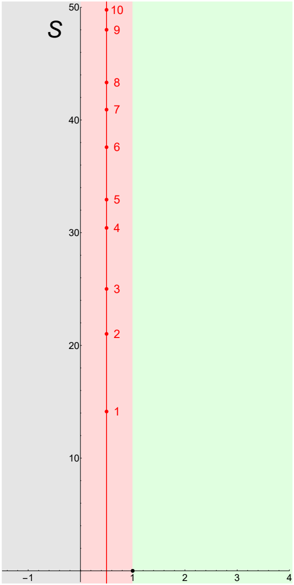
$$s \mapsto 4^{-s}$$

$$|s| = 4^{-1/2}$$



4th Dirichlet images of the first 10 zeta zeros

Dirichlet images of the zeta zeros

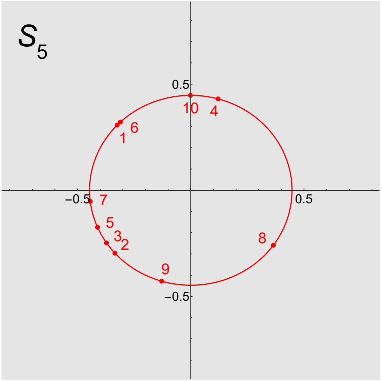


$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

n^{-s} is called the *nth Dirichlet image of s*

$$s \mapsto 5^{-s}$$

$$|s| = 5^{-1/2}$$



5th Dirichlet images of the first 10 zeta zeros

Numerical examples: $N = 5$

$a = 0.4 + 14\mathrm{i}$ $\quad \zeta(\rho_1) = 0$ $\quad \rho_1 = 0.5 + 14.134725141734\ldots\mathrm{i}$

n	q_n	$\lvert q_n/n^{-\rho_1} - 1\rvert$
2	$-0.658570722632\ldots + 0.257458025275\ldots\mathrm{i}$	$4.8927\ldots \cdot 10^{-8}$
3	$-0.568086335195\ldots - 0.103010905955\ldots\mathrm{i}$	$4.6606\ldots \cdot 10^{-8}$
4	$0.367430765659\ldots - 0.339108615925\ldots\mathrm{i}$	$5.8663\ldots \cdot 10^{-8}$
5	$-0.324829272639\ldots + 0.307385716024\ldots\mathrm{i}$	$9.3121\ldots \cdot 10^{-8}$

$a = 0.4 + 21\mathrm{i}$ $\quad \zeta(\rho_2) = 0$ $\quad \rho_2 = 0.5 + 21.0220396387715\ldots\mathrm{i}$

n	q_n	$\lvert q_n/n^{-\rho_2} - 1\rvert$
2	$-0.297469436794\ldots - 0.641491987361\ldots\mathrm{i}$	$3.9138\ldots \cdot 10^{-8}$
3	$-0.259863535982\ldots + 0.515562100991\ldots\mathrm{i}$	$3.5189\ldots \cdot 10^{-8}$
4	$-0.323023887992\ldots + 0.381648489607\ldots\mathrm{i}$	$1.7298\ldots \cdot 10^{-8}$
5	$-0.335079474007\ldots - 0.296178578044\ldots\mathrm{i}$	$9.9846\ldots \cdot 10^{-9}$

More numerical examples for $a = 0.4 + 14\mathrm{i}$

$$N = 100: \qquad \max_{1\le n\le N} \left|\frac{q_n}{n^{-\rho_1}} - 1\right| = 1.150\ldots \times 10^{-162}$$

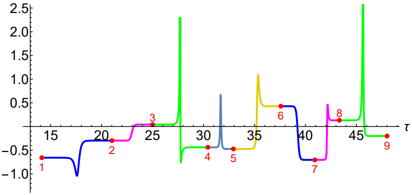
$$N = 150: \qquad \max_{1\le n\le N} \left|\frac{q_n}{n^{-\rho_1}} - 1\right| = 8.797\ldots \times 10^{-244}$$

$$N = 200: \qquad \max_{1\le n\le N} \left|\frac{q_n}{n^{-\rho_1}} - 1\right| = 6.939\ldots \times 10^{-325}$$

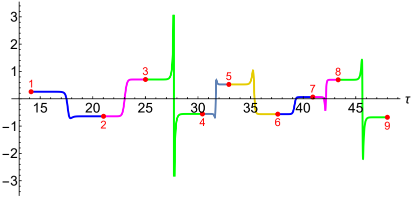
Away from zeros

$$N = 10 \qquad a = 0.4 + \mathrm{i}\tau \qquad \tau \text{ from } \mathrm{Im}(\rho_1) \text{ to } \mathrm{Im}(\rho_9)$$

$\mathrm{Re}(q_2)$



$\mathrm{Im}(q_2)$

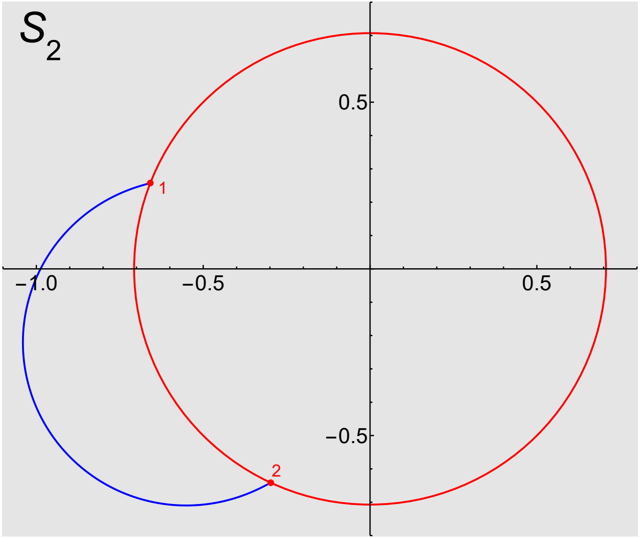


Conjecture B. *For all a except for a set of zero measure there is a zero ρ of the zeta function such that for all n*

$$q_n \to n^{-\rho} \text{ as } N \to \infty.$$

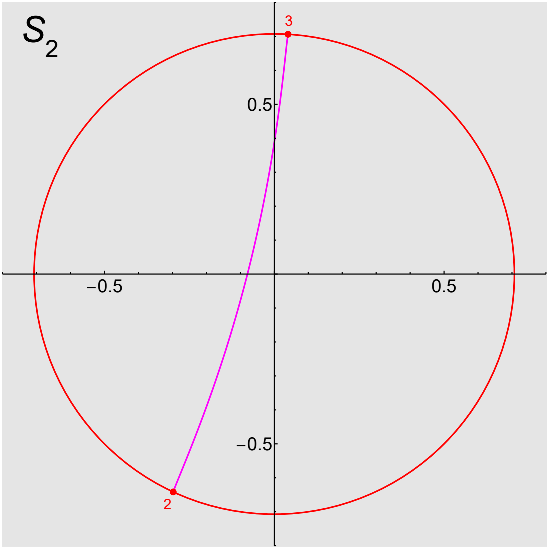
Away from zeros

q_2 for $N = 10$ $a = 0.4 + \mathrm{i}\tau$ τ from $\mathrm{Im}(\rho_1)$ to $\mathrm{Im}(\rho_2)$



Away from zeros

q_2 for $N = 10$ $\quad a = 0.4 + \mathrm{i}\tau$ $\quad \tau$ from $\mathrm{Im}(\rho_2)$ to $\mathrm{Im}(\rho_3)$

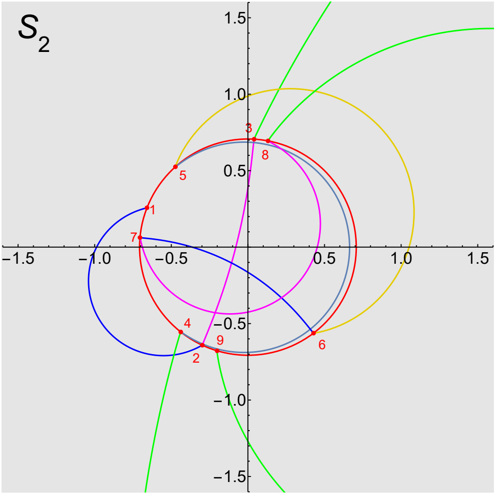


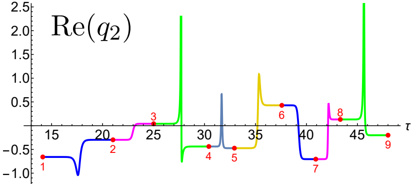
Away from zeros

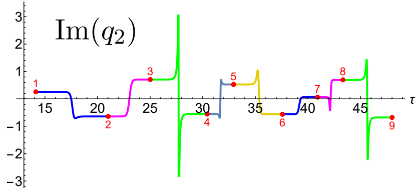
q_2 for $N = 10$

$a = 0.4 + \mathrm{i}\tau$

τ from $\mathrm{Im}(\rho_1)$ to $\mathrm{Im}(\rho_9)$



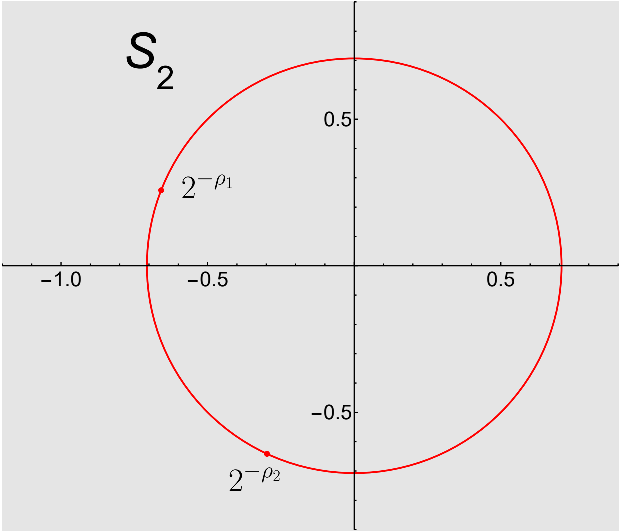




Away from zeros

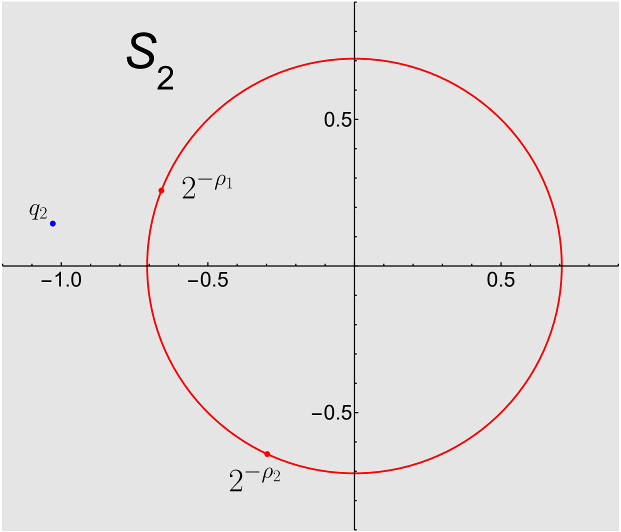
$\rho_1 = 0.5 + 14.1347\ldots\mathrm{i}$

$\rho_2 = 0.5 + 21.0220\ldots\mathrm{i}$



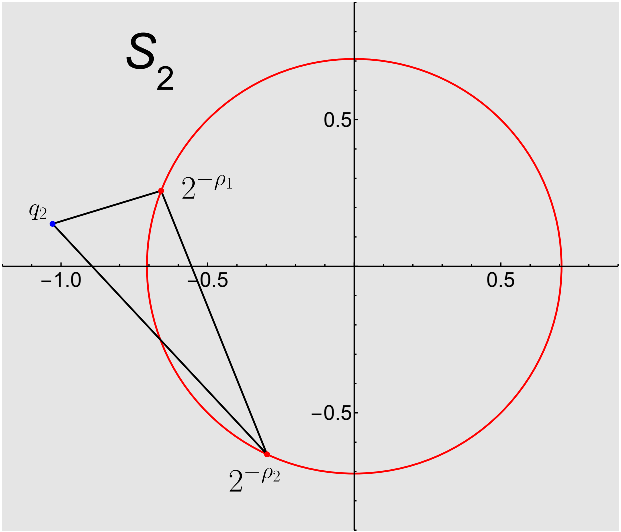
Away from zeros

$\rho_1 = 0.5 + 14.1347...\mathrm{i} \qquad a = 0.4 + 17.5\mathrm{i} \qquad \rho_2 = 0.5 + 21.0220...\mathrm{i}$



Away from zeros

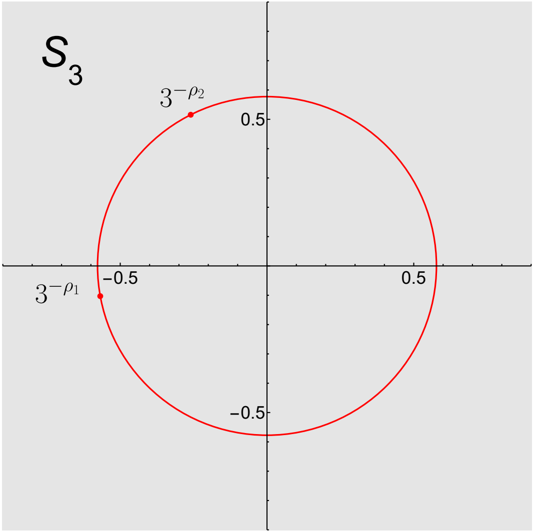
$$\rho_1 = 0.5 + 14.1347...\text{i} \qquad a = 0.4 + 17.5\text{i} \qquad \rho_2 = 0.5 + 21.0220...\text{i}$$



Away from zeros

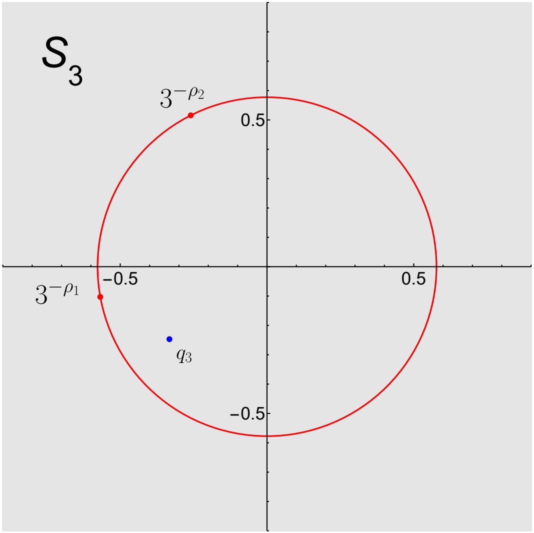
$\rho_1 = 0.5 + 14.1347...\mathrm{i}$

$\rho_2 = 0.5 + 21.0220...\mathrm{i}$



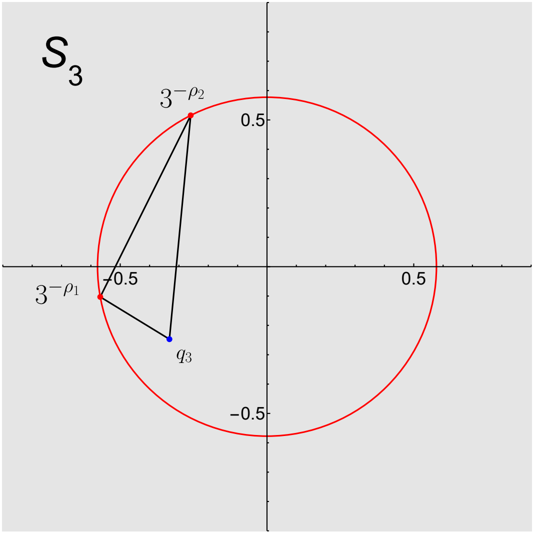
Away from zeros

$$\rho_1 = 0.5 + 14.1347...\text{i} \qquad a = 0.4 + 17.5\text{i} \qquad \rho_2 = 0.5 + 21.0220...\text{i}$$



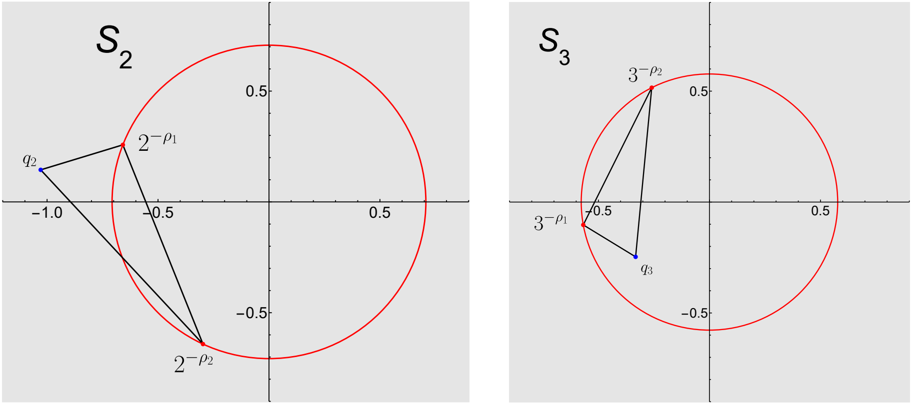
Away from zeros

$$\rho_1 = 0.5 + 14.1347...\mathrm{i} \qquad a = 0.4 + 17.5\mathrm{i} \qquad \rho_2 = 0.5 + 21.0220...\mathrm{i}$$

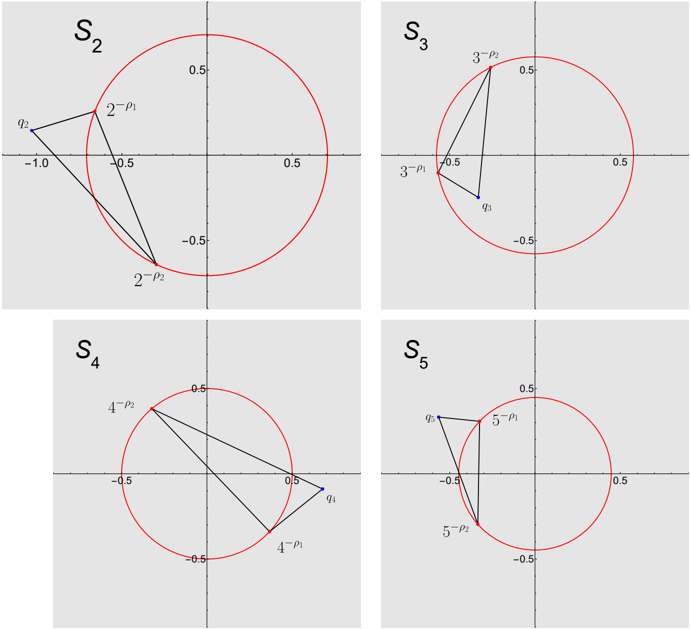


Away from zeros

$\rho_1 = 0.5 + 14.1347...\text{i} \qquad a = 0.4 + 17.5\text{i} \qquad \rho_2 = 0.5 + 21.0220...\text{i}$



Away from zeros



The law of (approximate) similarity

$$\frac{q_{n_2} - {n_2}^{-\rho_k}}{q_{n_1} - {n_1}^{-\rho_k}} \approx \frac{{n_2}^{-\rho_k} - {n_2}^{-\rho_{k+1}}}{{n_1}^{-\rho_k} - {n_1}^{-\rho_{k+1}}} \qquad 2 \leq n_1 < n_2 \leq N$$

$$n_1 = n \quad n_2 = n^2 \qquad \left(n^2\right)^{-\rho} = \left(n^{-\rho}\right)^2$$
$$n_1 = n \quad n_2 = n^3 \qquad \left(n^3\right)^{-\rho} = \left(n^{-\rho}\right)^3$$

$$n^{-\rho_k} + n^{-\rho_{k+1}} \approx \frac{q_n q_{n^2} - q_{n^3}}{q_n^2 - q_{n^2}} \qquad n^{-\rho_k} n^{-\rho_{k+1}} \approx \frac{q_{n^2}^2 - q_n q_{n^3}}{q_n^2 - q_{n^2}}$$

$n^{-\rho_k}$ and $n^{-\rho_{k+1}}$ are close to the two solutions of the equation

$$(q_n^2 - q_{n^2})r^2 - (q_n q_{n^2} - q_{n^3})r + (q_{n^2}^2 - q_n q_{n^3}) = 0$$

Numerical example

$$N = 50 \qquad n = 2 \qquad k = 1$$

$$\begin{aligned} q_2 &= -0.719... + 0.224...\mathrm{i} \\ q_4 &= 0.413... - 0.283...\mathrm{i} \\ q_8 &= -0.166... + 0.280...\mathrm{i} \end{aligned}$$

$$r_1 = -0.6585707... + 0.2574579...\mathrm{i}$$

$$r_2 = -0.2974694... - 0.6414919...\mathrm{i}$$

$$\frac{r_1}{2^{-\rho_1}} = 1 + (5.39119624... - 5.16170561...\mathrm{i}) \times 10^{-18}$$

$$\frac{r_2}{2^{-\rho_2}} = 1 + (0.449985441... - 6.150248171...\mathrm{i}) \times 10^{-17}$$