Approximation of the zeros of the Riemann zeta function by rational functions

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Abstract. We define rational functions $R_N(a, d_0,\ d_1,\ \dots,\ d_N)$ and demonstrate by numerical examples the following property. Let $d_0,\ d_1,\ \dots,\ d_N$ be equal respectively to the value of the Riemann zeta function and the values of its first N derivatives calculated at a. If a is not too far from a zero ρ of the zeta function, then the value of $R_N(a, d_0,\ d_1,\ \dots,\ d_N)$ is very close to ρ. For example, for $N = 10$ and $a = 0.6 + 14\mathrm{i}$ we have $|R_{10}(a, d_0,\ d_1,\ \dots,\ d_{10}) - \rho_1| < 10^{-14}$ where $\rho_1 = 0.5 + 14.13...\mathrm{i}$ is the first non-trivial zeta zero.

Also we define rational functions $R_N(a, d_0,\ d_1,\ \dots,\ d_N, n)$ which (under the same assumptions) have values which are very close to $n^{-\rho}$, that is, to the summands from the Dirichlet series for the zeta function calculated at its zero.

In the case when a is, say between two consecutive zeros, ρ_l and ρ_{l+1}, functions $R_N(a, d_0,\ d_1,\ \dots,\ d_N, n)$ approximate neither $n^{-\rho_l}$ nor $n^{-\rho_{l+1}}$; nevertheless, they allow us to approximate the sum $n^{-\rho_l} + n^{-\rho_{l+1}}$ and the product $n^{-\rho_l}\, n^{-\rho_{l+1}}$ and hence to calculate both $n^{-\rho_l}$ and $n^{-\rho_{l+1}}$ by solving corresponding quadratic equation.

Introduction

The celebrated *Riemann zeta function* $\zeta(s)$ can be defined for a complex number s with real part greater than 1 via a *Dirichlet series*,

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \tag{1}$$

and analytically extended to the whole complex plane. B. Riemann established a deep relationship between the zeros of this function and the prime numbers. Namely, he found an explicit expression for $\pi(x)$ – the number of primes below x – via these zeros.

The outstanding *Riemann Hypothesis* predicts that all the non-real zeros of the zeta function are lying on the *critical line* $\Re(s) = 1/2$. The hypothesis is equivalent to the assertion that

$$\pi(x) \quad = \quad \int_2^x \frac{\mathrm{d}y}{\ln(y)} + O(\ln(x)\sqrt{x}). \tag{2}$$

We address the following problem. Suppose that

$$d_k = \frac{\mathrm{d}^k}{\mathrm{d}s^k}\zeta(s)\bigg|_{s=a}, \quad k = 0, \dots, N \tag{3}$$

where number a is not far from a zero ρ of the zeta function. *How numbers $a, d_0,\ d_1,\ \dots,\ d_N$ could be used for calculating this ρ with high precision?*

1. Standard method

A straightforward way is to consider polynomial

$$T_N(s) = \sum_{k=0}^{N} \frac{d_k}{k!}(s-a)^k \tag{4}$$

(an initial fragment of Taylor series) and solve algebraic equation

$$T_N(s) = 0. \tag{5}$$

This approach has several drawbacks.

First, unless $N \le 4$, there is no "explicit" expression for the roots of this equation.

Second, we cannot analytically pinpoint which of the N roots of the equation is the desired approximation to ρ.

Third, ρ should be closer to a than the pole of the zeta function at $s = 1$.

2. Our method

We define

$$d_{m,k} = \frac{\mathrm{d}^k}{\mathrm{d}s^k}\zeta_m(s)\bigg|_{s=a} = \sum_{l=0}^{m} \binom{m}{l} \big(-\ln(n)\big)^{m-l} d_l, \tag{6}$$

$$T_{m,N}(s) = \sum_{k=0}^{N} \frac{d_{m,k}}{k!}(s-a)^k = b_{m,N,0} + \sum_{k=1}^{N} b_{m,N,k} s^k, \tag{7}$$

$$T_{m,N}(s_1, \dots, s_N) = b_{m,N,0} + \sum_{k=1}^{N} b_{m,N,k} s_k, \tag{8}$$

and solve linear system

$$T_{m,N}(s_1, \dots, s_N) = 0, \quad m = 1, \dots, N. \tag{9}$$

Clearly, $s_1, \dots, s_n$ are rational functions of $a, d_1, \dots, d_N$.

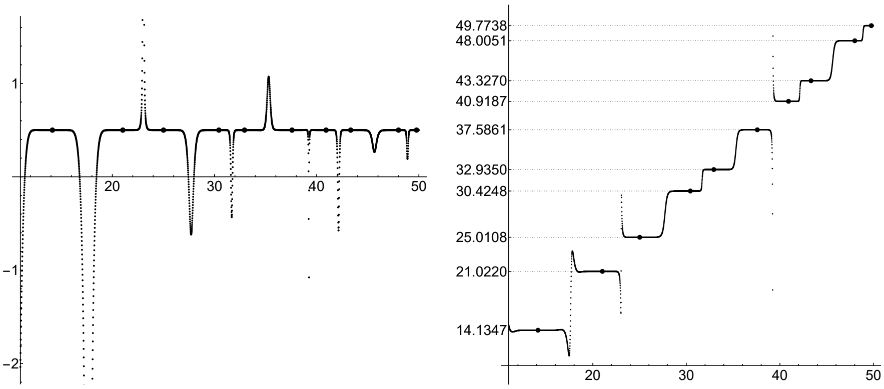

FIGURE 1. The small dots represent the real (on the left-hand plot) and the imaginary (on the right-hand plot) parts of s_1 from solutions of systems (9) for $N = 10$ and $a = 0.4 + \mathrm{i}\tau$, τ running from 10 to 50 with step 0.01. The ten larger dots on each plot represent the ten initial zeta zeros; the abscissas of the dots are equal to the imaginary parts of these zeros on both plots; the ordinates of the dots are equal respectively to the real and imaginary parts of the zeros.

Calculations demonstrate (see [1, 3]) that when a is not too far from a zero ρ of the zeta function, s_n is rather close to ρ_1. This is illustrated for $n = 1$ by Fig.1. We see that $\Re(s_1)$ is mainly equal to 1/2. As for $\Im(s_1)$, it looks almost as a step function with levels equal to the imaginary parts of the zeros of the zeta function.

Calculations suggest the following guess.

Conjecture A. For all a except for a set of zero measure there is a zero ρ of the zeta function such that for all k the value of s_k from the solution of system (9) tends to ρ^k as $N \to \infty$.

In order to define functions $R_N(a, d_0,\ d_1,\ \ldots,\ d_N, n)$ we replace polynomials (4) by Dirichlet polynomials

$$D_{m,N}(s) \quad = \quad \sum_{n=1}^{N} c_{m,N,n} n^{-s} \tag{10}$$

having the same initial Taylor expansion as $\zeta_m(s)$. Polynomials (8) are replaced by linear polynomials

$$Q_{m,N}(q_2, \ldots, q_N) \quad = \quad c_{m,N,1} + \sum_{n=2}^{N} c_{m,N,n} q_n \tag{11}$$

and $R_N(a, d_0,\ d_1,\ \ldots,\ d_N, n)$ is defined as the value of q_n from the solution of system

$$Q_{m,N}(q_2, \ldots, q_N) = 0, \quad m = 1, \ldots, N-1. \tag{12}$$

So defined q_n approximate very well $n^{-\rho}$ when a is not too far from a zero ρ of the zeta function (see [2, 3]). This suggest the following guess.

Conjecture B. For all a except for a set of zero measure there is a zero ρ of the zeta function such that for all $n > 1$ the value of q_n from the solution of system (12) tends to ρ^n as $N \to \infty$.

In the case when a is, say between two consecutive zeros, ρ_l and ρ_{l+1}, the value of q_n is close neither to $n^{-\rho_l}$ nor to $n^{-\rho_{l+1}}$, but

$$\frac{q_n q_{n^2} - q_{n^3}}{q_n^2 - q_{n^2}} \approx n^{-\rho_l} + n^{-\rho_{l+1}}, \qquad \frac{q_{n^2}^2 - q_n q_{n^3}}{q_n^2 - q_{n^2}} \approx n^{-\rho_l} n^{-\rho_{l+1}}, \tag{13}$$

and we can calculate approximations to both $n^{-\rho_l}$ and to $n^{-\rho_{l+1}}$ by solving corresponding quadratic equation.

References

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