On integrability of the resonant cases of the generalized Lotka–Volterra system

Victor F. Edneral and Alexander B. Aranson

Abstract. The paper discusses a possible connection between local integrability near stationary points and global integrability of an autonomous two-dimensional polynomial ODE system. As an example, we use the resonance case of the generalized Lotka–Volterra system. We parametrized its right-hand sides as quadratic polynomials with resonance linear part. The conditions of local integrability near stationary points are written as systems of algebraic equations in the parameters of the system. We solve these systems. It is established that for the values of the parameters obtained in this way, the system of ODEs under consideration turns out to be integrable. Thus, we can speak of a heuristic approach that allows us to determine cases of ODE integrability a priori.

Introduction

We use an approach based on local analysis. It uses the resonant normal form computed near stationary points [1]. In the paper [2] we proposed a method for searching for integrable cases based on determining the parameter values for which the dynamical system is locally integrable at all stationary points simultaneously. Because at regular points, local integrability always holds, so such a requirement is equivalent to the requirement of local integrability at every point of the domain under consideration.

Note that the integrability of an autonomous planar system implies the solvability of the system in quadratures.

Problem

We will check our method on the example of a resonance case of the generalized Lotka–Volterra system

$$\begin{array}{rl} \dot{x} = & Mx + a_1 x^2 + a_2 x\, y + a_3 y^2, \\ \dot{y} = & -\,y + b_1 x^2 + b_2 x\, y + b_3 y^2, \end{array} \qquad (1)$$

here x and y are functions in time and parameters $a_1, a_2, a_3, b_1, b_2, b_3$ are real. M is non-negative integer. Each value of M corresponds to the resonance $M : 1$.

The problem is to construct the first integrals of system (1).

Method

The main task of the method under discussion is to find conditions on the parameters of the system under which the system is locally integrable near its stationary points. Local integrability means the presence of a sufficient number (one for an autonomous flat system) of local integrals. Local integrals may be different for different points of this region of the phase space, but in our opinion for the existence of a global integral, the enough number of local integrals (one in our case) must exist for in each point simultaneously. This condition is not satisfied for arbitrary parameter values. In the book[1] the algebraic condition of local integrability is written out. This is the so-called **A** condition. It is satisfied at all regular points, but it is nontrivial in resonance cases at stationary points.

Firstly we look for sets of parameters under which the condition **A** is satisfied at the stationary point of the system (1) at the origin. We solve the corresponding systems of algebraic equations with respect to the parameters $a_1, a_2, a_3, b_1, b_2, b_3$ and check the integrability at other stationary points for each found set of parameters. Received parameter sets are good candidates for the existence of a single function for all points - the first integral. These integrals are sought by one method or another. We did this procedure for 3 resonances $M : 1, M = 1, 2, 3$.

Condition of Local Integrability

In our case the condition of local integrability **A** is some infinite sequence of polynomial equations with respect to the coefficients of the system. Each of the stationary points has its own system of equations. The joined system should be solved. Another tactic involves solving system at stationary point in the origin and checking the corresponding solutions with the condition **A** at other points of the phase plane. Recall that the normal form has a non-trivial form in the resonant case only.

Calculation of the normal form is an iterative process, we do it step by step. Computing each $M + 1$ normal form order adds one equation to the condition. Condition **A** is an infinite system of equations, we have to work with a finite (truncated) condition. But as a result of many calculations in various systems of

ODEs, we noticed a remarkable fact. After a certain order, adding new equations to condition **A** ceases to affect the solutions of the system, i.e. several of its lower polynomials form the basis of the entire infinite ideal. For the system under consideration, it suffices to consider the first 3 equations. We generated them by the package [3].

For a 1:1 resonance, this is the truncated **A** system at the origin. It has been experimentally established that adding further equations does not change its solution

$$
\begin{aligned}
& a_1a_2 - b_2b_3 = 0, \\
& -a_3b_2(-6a_1^2 + 9a_1b_2 + 14b_1b_3 + 6b_2^2) + 9a_2^2(a_1b_2 + b_1b_3) + a_2(14a_1a_3b_1 - \\
& \quad 3b_3(2b_1b_3 + 3b_2^2)) + 6a_2^3b_1 = 0, \\
& 432a_1^4a_2a_3 + 36a_1^3(54a_2^3 + 18a_2^2b_3 - 61a_2a_3b_2 - 18a_3b_2b_3) - \\
& \quad 6a_1^2(162a_2^3b_2 + a_2^2(131a_3b_1 - 162b_2b_3) + 3a_2a_3(106b_1b_3 + 75b_2^2) + \\
& \quad 2a_3b_2(194a_3b_1 - 381b_2b_3)) + a_1(3708a_2^4b_1 - 108a_2^3(33b_2^2 - 38b_1b_3) - \\
& \quad 3a_2^2b_1(5299a_3b_2 + 1524b_3^2) - 4a_2(868a_3^2b_1^2 - 981a_3b_2^3 + 81b_3^2(3b_2^2 - 2b_1b_3)) + \\
& \quad 36b_2(142a_3^2b_1b_2 + a_3b_3(53b_1b_3 - 114b_2^2) - 18b_2b_3^3)) - 1782a_2^4b_1b_2 \\
& \quad -6a_2^3b_1(523a_3b_1 + 654b_2b_3) + 18a_2^2b_3(-284a_3b_1^2 + 75b_1b_2b_3 + 198b_2^3) + \\
& \quad 3a_2(a_3(776b_1^2b_3^2 + 5299b_1b_2^2b_3 + 594b_2^4) + 12b_2b_3^2(61b_1b_3 + 27b_2^2)) + \\
& \quad 2b_2(a_3^2b_1(1736b_1b_3 + 1569b_2^2) + 3a_3b_2b_3(131b_1b_3 - 618b_2^2) - \\
& \quad 108b_3^3(2b_1b_3 + 9b_2^2)) = 0.
\end{aligned}
\tag{2}
$$

Equations of a similar form were obtained for resonances 1 : 2 and 1 : 3 also.

Results

The MATHEMATICA-11 system received 11 rational solutions of system (2). Some of them are a consequence of others. 7 solutions turned out to be independent:

$$
\begin{aligned}
& 1)\{a_1 \to -\tfrac{b_2}{2}, b_3 \to -\tfrac{a_2}{2}\}; \\
& 2)\{a_3 \to \tfrac{a_2^3b_1}{b_2^3}, b_3 \to \tfrac{a_1a_2}{b_2}\}; \\
& 3)\{a_1 \to 2b_2, a_3 \to \tfrac{a_2b_2}{b_1}, b_3 \to 2a_2\}; \\
& 4)\{a_1 \to 2b_2, a_3 \to 0, b_1 \to 0, b_3 \to 2a_2\}; \\
& 5)\{a_2 \to 0, b_2 \to 0\}; \\
& 6)\{a_1 \to 0, b_1 \to 0, b_2 \to 0\}; \\
& 7)\{a_1 \to 2b_2, a2 \to 0, b_1 \to 0, b_3 \to 0\}.
\end{aligned}
\tag{3}
$$

At these sets of parameters we checked the integrability condition at other stationary points of system (2).

The corresponding cases of system (1) look like this:

$$\begin{array}{lll}
1)\dot{x} = & x - \frac{1}{2}b_2x^2 + a_2xy + a_3y^2, & \dot{y} = -y + b_1x^2 + b_2xy - \frac{1}{2}a_2y^2;\\
2)\dot{x} = & x + a_1x^2 + a_2xy + \frac{a_2^3b_1}{b_2^3}y^2, & \dot{y} = -y + b_1x^2 + b_2xy + \frac{a_1a_2}{b_2}y^2;\\
3)\dot{x} = & x + 2b_2x^2 + a_2xy + \frac{a_2b_2}{b_1}y^2, & \dot{y} = -y + b_1x^2 + b_2xy + 2a_2y^2;\\
4)\dot{x} = & x + 2b_2x^2 + a_2xy, & \dot{y} = -y + b_2xy + 2a_2y^2;\\
5)\dot{x} = & x + a_1x^2 + a_3y^2, & \dot{y} = -y + b_1x^2 + b_3y^2;\\
6)\dot{x} = & x + a_2xy + a_3y^2, & \dot{y} = -y + b_3y^2;\\
7)\dot{x} = & x + 2b_2x^2 + a_3y^2, & \dot{y} = -y + b_2xy.
\end{array} \quad (4)$$

Cases 1), 4),6) and 7) have been integrated by the MATHEMATICA-11 solver. 2) and 5) were integrated by the Darboux method. At this moment we could not integrate case 3). For the 1 : 2 and 1 : 3 resonances, we also managed to calculate the integrals for almost all predicted cases.

We then combined the conditions for the three resonances and tried to integrate the corresponding cases of the general (non-resonant) form. Systems with coefficients thus obtained have the form

$$\begin{array}{lll}
1) & \dot{x} = \alpha x + a_1x^2, & \dot{y} = -y + b_1x^2 + b_3y^2;\\
2) & \dot{x} = \alpha x + a_1x^2, & \dot{y} = -y + b_1x^2 + b_2xy;\\
3) & \dot{x} = \alpha x + a_2xy + a_3y^2, & \dot{y} = -y + b_3y^2;\\
4) & \dot{x} = \alpha x + 2b_2x^2 + a_3y^2, & \dot{y} = -y + b_2xy;\\
5) & \dot{x} = \alpha x + a_2xy, & \dot{y} = -y + b_2xy;\\
6) & \dot{x} = \alpha x + a_2xy + a_3y^2, & \dot{y} = -y;\\
7) & \dot{x} = \alpha x + a_2xy + a_3y^2, & \dot{y} = -y - a_2y^2/2;\\
8) & \dot{x} = \alpha x + b_2x^2 + a_2xy, & \dot{y} = -y + b_2xy + a_2y^2;\\
9) & \dot{x} = \alpha x + a_2xy + a_3y^2, & \dot{y} = -y + a_2y^2;\\
10) & \dot{x} = \alpha x + a_2xy, & \dot{y} = -y + b_1x^2 + 2a_2y^2;\\
11) & \dot{x} = \alpha x + a_2xy + a_3y^2, & \dot{y} = -y + 2a_2y^2.
\end{array} \quad (5)$$

α here is voluntary parameter. That is, we have moved away from the resonant limitation.

All systems (5) are integrable. So we have an algorithm that can predict integrable cases.

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Victor F. Edneral
Skobeltsyn Institute of Nuclear Physics of Lomonosov Moscow State University
1(2) Leninskie gory, Moscow, 119991, Russian Federation
e-mail: `edneral@theory.sinp.msu.ru`

Alexander B. Aranson
quit Scientific Research Institute of Long-Range Radio Communication
Moscow, Russian Federation
e-mail: `aboar@yandex.ru`