On the integrability of the resonant case of the generalized Lotka–Volterra system

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Integrability

For the autonomous ODE system

$$\frac{d\,x_i}{d\,t} = \phi_i(x_1,\ldots,x_n), \quad i = 1,\ldots,n,$$

the total derivative of the first integral $I_k(x_1,\ldots,x_n)$ with respect to time is equal zero along the trajectory in the phase space of that system

$$\left.\frac{d\,I_k(x_1,\ldots,x_n)}{d\,t}\right|_{\frac{d\,x_i}{d\,t}=\phi_i(x_1,\ldots,x_n)} = 0, \quad k = 1,\ldots,m.$$

- The system can have m such integrals. System is called integrable if it has enough numbers of the first integrals.
- For integrability of an autonomous two-dimensional system, it is enough to have a single integral.

Example

- Let us see the harmonic oscillator

$$\ddot{x}(t) + \omega_0^2 \cdot x(t) = 0$$

- This is equivalent to the notation

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = -\omega_0^2 \cdot x(t). \end{cases} \quad (1)$$

- The first integral here is

$$I(x(t), y(t)) = x^2(t) + y^2(t)/\omega_0^2.$$

- Its total derivation in time is

$$\frac{d\, I(x(t), y(t))}{d\, t} = 2x(t)\dot{x}(t) + 2y(t)\dot{y}(t)/\omega_0^2 = 2x(t)y(t) - 2x(t)y(t) = 0$$

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due to (1).

Solution

Of constancy of the first integral

$$I(x(t), y(t)) = C_1^2,$$

we have

$$y(t) = \sqrt{\omega_0^2 \cdot (C_1^2 - x^2(t))}.$$

By substituting $y(t)$ in system (1) we get

$$\frac{dx(t)}{dt} = \sqrt{\omega_0^2 \cdot (C_1^2 - x^2(t))},$$

or

$$\frac{dx(t)}{\sqrt{C_1 - x^2(t)}} = \omega_0 \cdot dt, \quad \text{i.e. } \arcsin(x(t)/C_1) = \omega_0 \cdot t + C_2.$$

Finally we get $x(t) = C_1 \cdot \sin(\omega_0 \cdot t + C_2)$.

- Integrability is an important property of the system. In particular, if a system is integrable then it is solvable by quadrature.
- The knowledge of the integrals is important also at the investigation of a phase portrait, for the creation of symplectic integration schemes e.t.c.

Task

- Generally, integrability is a rare property.
- But the system may depend on parameters.
- Our task here is to find the values of system parameters at which the system is integrable.
- To solve this problem, we try to use the local analysis. It studies the behavior of the system in a neighborhood of a point in phase space.

Local integrability

We consider an autonomous system of ordinary differential equations

$$\frac{d\,x_i}{d\,t} \stackrel{\mathrm{def}}{=} \dot{x}_i = \phi_i(X), \quad i = 1, \ldots, n, \tag{2}$$

where $X = (x_1, \ldots, x_n) \in \mathbb{C}^n$ and $\phi_i(X)$ are polynomials.

In a neighborhood of the point $X = X^0$, the system (2) is *locally integrable* if it has there sufficient number m of independent first integrals of the form

$$I_k(X) = \frac{a_k(X)}{b_k(X)}, \quad k = 1, \ldots, m,$$

where functions $a_k(X)$ and $b_k(X)$ are analytic in a neighborhood of this point. Such $I_k(X)$ are called the formal integrals.

Generally speaking the formal integrals are not related to the first integral of the system functionally.

Resonance normal form

- The resonance normal form was introduced by Poincaré for the investigation of systems of nonlinear ordinary differential equations. It is based on the maximal simplification of the right-hand sides of these equations by invertible transformations.
- The normal form approach was developed in works of G.D. Birkhoff, T.M. Cherry, A. Deprit, F.G. Gustavson, C.L. Siegel, J. Moser, A.D. Bruno and others. This technique is based on the Local Analysis method by Prof. Bruno [Bruno 1971, 1972, 1979, 1989].

Multi-index notation

Let's suppose that we treat the reduced to a diagonal polynomial system near a stationary point at the origin and rewrite this n-dimension system in the terms

$$\dot{x}_i = \lambda_i x_i + x_i \sum_{\mathbf{q}\in\mathcal{N}_i} f_{i,\mathbf{q}} \mathbf{x}^{\mathbf{q}}, \quad i = 1, \ldots, n\,, \tag{3}$$

where we use the multi-index notation

$$\mathbf{x}^{\mathbf{q}} \equiv \prod_{j=1}^{n} {x_j}^{q_j}$$

with the power exponent vector $\mathbf{q} = (q_1, \ldots, q_n)$
Here the sets:

$$\mathcal{N}_i = \{\mathbf{q} \in \mathbb{Z}^n : q_i \geq -1 \text{ and } q_j \geq 0\,,\ \text{if } j \neq i\,, \quad j = 1, \ldots, n\}\,,$$

because the factor x_i has been moved out of the sum in (3).

Above we assumed that the origin is a stationary point. Otherwise, we must use shift.

We supposed also that the linear part of the system have been reduced to the diagonal form. Note that there exist general formulas for the Jordan case of the linear part but we do not need to use the general case in this report.

Normal form

The normalization is done with a near-identity reversible transformation:

$$x_i \;=\; z_i + z_i \sum_{\mathbf{q}\in\mathcal{N}_i} h_{i,\mathbf{q}}\mathbf{z}^{\mathbf{q}}, \quad i = 1, \ldots, n \tag{4}$$

after which we have system (3) in the normal form:

$$\dot{z}_i = \lambda_i z_i + z_i \sum_{\substack{\langle \mathbf{q}, \mathbf{L}\rangle = 0 \\ \mathbf{q}\in\mathcal{N}_i}} g_{i,\mathbf{q}}\mathbf{z}^{\mathbf{q}}, \quad i = 1, \ldots, n, \tag{5}$$

where $\mathbf{L} = \{\lambda_1, \ldots, \lambda_n\}$ is the vector of eigenvalues.

Theorem (Bruno 1971)

There exists a formal change (4) reducing (3) to its normal form (5).

Note, the normalization (4) does not change the linear part of the system.

Resonance terms

The important difference between (3) and (5) is a restriction on the range of the summation, which is defined by the equation:

$$\langle \mathbf{q}, \mathbf{L} \rangle = \sum_{j=1}^{n} q_j \lambda_j = 0 \,. \qquad (6)$$

I.e. the summation in the normal form (5) contains only terms, for which (6) is valid. They are called resonance terms.

Note, if the eigenvalues are not comparable then condition (6) is never valid at any components of the vector **q**, because they are integer.

Calculation of the normal form

The h and g coefficients in (4) and (5) are found by using the recurrent formula:

$$g_{i,\mathbf{q}} + \langle \mathbf{q}, \mathbf{L} \rangle \cdot h_{i,\mathbf{q}} = - \sum_{j=1}^{n} \sum_{\substack{\mathbf{p}+\mathbf{r}=\mathbf{q} \\ \mathbf{p},\mathbf{r} \in \bigcup_i \mathcal{N}_i \\ \mathbf{q} \in \mathcal{N}_i}} (p_j + \delta_{ij}) \cdot h_{i,\mathbf{p}} \cdot g_{j,\mathbf{r}} + \tilde{\Phi}_{i,\mathbf{q}} \,, \quad (7)$$

For this calculation we have two programs.

- in LISP [Edneral, Khrustalev 1992]
- in the high-level language of the MATHEMATICA system [Edneral, Khanin 2002].

Conditions **A and** ω

There are two conditions

- Condition **A**. In the normal form (5)

$$g_j(Z) = \lambda_j a(Z) + \bar{\lambda}_j b(Z), \quad j = 1, \dots, n, \tag{8}$$

  where $a(Z)$ and $b(Z)$ are some formal power series.
- Condition ω (on small divisors) [Bruno 1971]. It is fulfilled for almost all vectors **L**. At least it is satisfied at rational eigenvalues.

Theorem (Bruno 1971)

If vector **L** *satisfies Condition* ω *and the normal form* (5) *satisfies Condition* **A** *then the normalizing transformation (4) converges.*

Formal integral

Let us rewrite the normalized equation (5) as

$$\dot{z}_i = \lambda_i z_i + z_i g_i(Z),$$

or

$$\frac{d}{dt} \log z_i = \lambda_i + g_i(Z),$$

where $g_i(Z)$ is the re-designate sum.

Consider the case of a $N : M$ resonance in the two-dimension system. The condition **A** is

$$g_1(Z) = \lambda_1 a(Z) + \bar{\lambda}_1 b(Z), \quad g_2(Z) = \lambda_2 a(Z) + \bar{\lambda}_2 b(Z).$$

The eigenvalues satisfy the ratio $N \cdot \lambda_1 = -M \cdot \lambda_2$,
therefore $N \cdot g_1(Z) = -M \cdot g_2(Z)$.

I.e. the normalized system can be rewritten as

$$N \times \left| \frac{d\,\log(z_1)}{d\,t} = \lambda_1 + g_1(Z)\,, \quad -M \times \right| \frac{d\,log(z_2)}{d\,t} = \lambda_2 + g_2(Z)\,.$$

So, $\frac{d\,\log(z_1^N \cdot z_2^M)}{d\,t} = 0$ or $z_1^N \cdot z_2^M = const$

and $z_1^N \cdot z_2^M$ is the formal integral.

Near a stationary point the condition **A**:

- Ensures convergence;
- Provides the local integrability;
- Isolates the periodic orbits if the eigenvalues are pure imagine.

Note, that the local integrability condition above works in the multidimensional case also.

Hypothesis

- Initially, we experimentally discovered that if condition of the local integrability **A** is satisfied at all stationary points of some domain of the phase space, then we can find the first integral of a two-dimensional autonomous polynomial system [Edneral, Romanovsky, report at the ACA 2018].
- By integrability we mean the existence in some domain of the phase space of a differentiable function that is constant along the solutions of the system.
- Note, that the system locally integrable at all regular points. Thus, the requirement of local integrability at stationary points can be expanded as a conjecture

Hypothesis

Local integrability in a neighborhood of each point of some domain of the phase space leads to the existence of the first integral in this domain.

Integrability Condition as an Algebraic System

- The condition of local integrability the condition **A** is an infinite system of algebraic equations in the system parameters. But over a number field it is equivalent to a finite system.
- We can calculate the lowest equations of the the condition **A** by the CA program. We suppose that we have calculated enough numbers of the equations. On experiments an expansion of that system does not change its solutions. In any case, we obtain a necessary condition for integrability.
- So, we will have the condition of integrability as a finite system of algebraic equations in the system parameters.

Problem

We will check our method on the example of a resonance case of the generalized Lotka–Volterra system

$$\begin{aligned} \dot{x} &= \quad M\,x + a_1 x^2 + a_2 x\,y + a_3 y^2, \\ \dot{y} &= \quad -y + b_1 x^2 + b_2 x\,y + b_3 y^2, \end{aligned} \tag{9}$$

here x and y are functions in time and parameters $a_1, a_2, a_3, b_1, b_2, b_3$ are real. M is non-negative integer. Each value of M corresponds to the resonance $M : 1$.

The problem is to find the values of parameters at which the system has the first integral.

Condition of the Local Integrability as Algebraic Equations

For a 1:1 resonance, this is the truncated **A** system at the origin. It has been experimentally established that adding further equations does not change its solution

$$a_1 a_2 - b_2 b_3 = 0,$$

$$-a_3 b_2(-6a_1^2 + 9a_1 b_2 + 14b_1 b_3 + 6b_2^2) + 9a_2^2(a_1 b_2 + b_1 b_3) + a_2(14a_1 a_3 b_1 - 3b_3(2b_1 b_3 + 3b_2^2)) + 6a_2^3 b_1 = 0,$$

$$432a_1^4 a_2 a_3 + 36a_1^3(54a_2^3 + 18a_2^2 b_3 - 61a_2 a_3 b_2 - 18a_3 b_2 b_3) - 6a_1^2(162a_2^3 b_2 + a_2^2(131a_3 b_1 - 162b_2 b_3) + 3a_2 a_3(106b_1 b_3 + 75b_2^2) + 2a_3 b_2(194a_3 b_1 - 381b_2 b_3)) + a_1(3708a_2^4 b_1 - 108a_2^3(33b_2^2 - 38b_1 b_3) - 3a_2^2 b_1(5299a_3 b_2 + 1524b_3^2) - 4a_2(868a_3^2 b_1^2 - 981a_3 b_2^3 + 81b_3^2(3b_2^2 - 2b_1 b_3)) + 36b_2(142a_3^2 b_1 b_2 + a_3 b_3(53b_1 b_3 - 114b_2^2) - 18b_2 b_3^3)) - 1782a_2^4 b_1 b_2 - 6a_2^3 b_1(523a_3 b_1 + 654b_2 b_3) + 18a_2^2 b_3(-284a_3 b_1^2 + 75b_1 b_2 b_3 + 198b_2^3) + 3a_2(a_3(776b_1^2 b_3^2 + 5299b_1 b_2^2 b_3 + 594b_2^4) + 12b_2 b_3^2(61b_1 b_3 + 27b_2^2)) + 2b_2(a_3^2 b_1(1736b_1 b_3 + 1569b_2^2) + 3a_3 b_2 b_3(131b_1 b_3 - 618b_2^2) - 108b_3^3(2b_1 b_3 + 9b_2^2)) = 0.$$

Equations of a similar form were obtained for resonances $1:2$ and $1:3$ also.

Solutions of the Condition

The MATHEMATICA-11 system received 11 rational solutions of the system above. Some of them are a consequence of others. 7 solutions turned out to be independent:

$$
\begin{array}{ll}
1) & \left\{a_1 \to -\frac{b_2}{2}, b_3 \to -\frac{a_2}{2}\right\}, \\
 & \left\{a_1 \to 0, b_2 \to 0, b_3 \to -\frac{a_2}{2}\right\} \quad \text{is secuence of 1)}, \\
2) & \left\{a_3 \to \frac{a_2^3 b_1}{b_2^3}, b_3 \to \frac{a_1 a_2}{b_2}\right\}, \\
 & \left\{a_1 \to 2b_2, a_3 \to \frac{a_2^3 b_1}{b_2^3}, b_3 \to 2a_2\right\} \quad \text{is secuence of 2)}, \\
3) & \left\{a_1 \to 2b_2, a_3 \to \frac{a_2 b_2}{b_1}, b_3 \to 2a_2\right\}, \\
 & \{a_1 \to 0, a_3 \to 0, b_2 \to 0, b_3 \to 2a_2\} \quad \text{is secuence of 3)}, \\
4) & \{a_1 \to 2b_2, a_3 \to 0, b_1 \to 0, b_3 \to 2a_2\}, \\
5) & \{a_2 \to 0, b_2 \to 0\}, \\
6) & \{a_1 \to 0, b_1 \to 0, b_2 \to 0\}, \\
 & \{a_1 \to 0, b_1 \to 0, b_2 \to 0, b_3 \to 2a_2\} \quad \text{is secuence of 6)}, \\
7) & \{a_1 \to 2b_2, a_2 \to 0, b_1 \to 0, b_3 \to 0\}.
\end{array}
$$

At these sets of parameters we checked the integrability condition at other stationary points of the system above.

Calculation of the first integrals

An autonomous second order system can be rewritten as a non-autonomous first order equation. Let

$$\frac{d\,x(t)}{d\,t} = P(x(t), y(t)), \quad \frac{d\,y(t)}{d\,t} = Q(x(t), y(t)).$$

We divided the left and right hand sides of the system equations into each other. In result we have the first-order differential equation for $x(y)$ or $y(x)$

$$\frac{d\,x(y)}{d\,y} = \frac{P(x(y), y)}{Q(x(y), y)} \quad \text{or} \quad \frac{d\,y(x)}{d\,x} = \frac{Q(x, y(x))}{P(x, y(x))}.$$

Then we solved them by the MATHEMATICA-11 solver and sometimes got the solution $y(x)$ (or $x(y)$). After that we calculated the integral from this solution by extracting the integration constant $C[1]$.

Integrals

We tried to calculate the integrals for these cases:

1) $\dot{x} = x - \frac{1}{2}b_2x^2 + a_2xy + a_3y^2, \quad \dot{y} = -y + b_1x^2 + b_2xy - \frac{1}{2}a_2y^2,$

$I_1 = 2b_1x^3 + 3b_2x^2y - 2a_3y^3 - 3xy(2 + a_2y);$

2) $\dot{x} = x + a_1x^2 + a_2xy + \frac{a_2^3b_1}{b_2^3}y^2, \quad \dot{y} = -y + b_1x^2 + b_2xy + \frac{a_1a_2}{b_2}y^2,$

$$I_2 = \left(\frac{(a_2b_1 - a_1b_2)(a_2y - b_2x) + b_2^2}{b_2^2}\right)^{\frac{b_2(2a_1+b_2)}{a_2b_1-a_1b_2}+1} \times$$
$$\Big((b_2(a_1+b_2) + a_2b_1)\left(b_2y\left(x\left(2a_2b_1+b_2^2\right)(b_2(b_2-a_1)+a_2b_1) - 2a_2b_1b_2\right) + a_2^2b_1y^2\left(2a_2b_1+b_2^2\right) + b_1b_2^2x\left(x\left(2a_2b_1+b_2^2\right)+2b_2\right)\right) + 2b_1b_2^4\Big);$$

3) $\dot{x} = x + 2b_2x^2 + a_2xy + \frac{a_2b_2}{b_1}y^2, \quad \dot{y} = -y + b_1x^2 + b_2xy + 2a_2y^2,$

Memory is exosted;

4) $\dot{x} = x + 2b_2x^2 + a_2xy, \quad \dot{y} = -y + b_2xy + 2a_2y^2,$

$$I_4 = \frac{216a_2^3y^3 - 648a_2^2b_2xy^2 - 324a_2^2y^2 + 648a_2b_2^2x^2y + 648a_2b_2xy + 162a_2y - 216b_2^3x^3 - 324b_2^2x^2 - 162b_2x - 27}{x^2y^2};$$

5) $\dot{x} = x + a_1x^2 + a_3y^2, \quad \dot{y} = -y + b_1x^2 + b_3y^2,$

$$I_5 = b_1(a_3b_1 - a_1b_3)\times$$
$$\int \frac{(b_1x^2 + b_3y^2 - y)\,dx}{x\left(-x\left(a_1^2 + b_1x(a_1b_3 - a_3b_1) + b_1b_3\right) - 2a_1\right) - y^2\left(b_3x(a_1b_3 - a_3b_1) + a_1a_3 + b_3^2\right) + y(2b_3 - x(a_1x+3)(a_3b_1 - a_1b_3)) + a_3y^3(a_1b_3 - a_3b_1) - 1};$$

6) $\dot{x} = x + a_2xy + a_3y^2, \quad \dot{y} = -y + b_3y^2,$

$$I_6 = \frac{(1 - b_3y)^{-\frac{a_2}{b_3}}\left(a_2^3xy + a_2^2a_3y^2 + a_2a_3b_3y^2 - 2a_2a_3y - a_2b_3^2xy - 2a_3b_3y + 2a_3\right)}{b_3y - 1};$$

7)$\dot{x} = x + 2b_2x^2 + a_3y^2, \quad \dot{y} = -y + b_2xy,$

$$I_7 = \frac{y^2\left(a_3y^2 + 3x\right)^2}{\left(a_3b_2y^2 + 2b_2x + 1\right)^3}.$$

Cases 1), 4), 6) and 7) have been integrated by the MATHEMATICA-11 solver. 2) and 5) were integrated by the Darboux method. At this moment we could not integrate case 3).

For the $1:2$ and $1:3$ resonances, we also managed to calculate the integrals for almost all predicted cases.

General system

The next step was to combine algebraic equations for local integrability conditions for all three calculated resonances. The solutions of the resulting system of 9 equations can be tried to predict the integrable cases of the general system

$$\begin{aligned} \dot{x} = &\quad \alpha\, x + a_1 x^2 + a_2 x\, y + a_3 y^2, \\ \dot{y} = &\quad -y + b_1 x^2 + b_2 x\, y + b_3 y^2, \end{aligned} \tag{10}$$

α here is an arbitrary parameter.

Solutions of the Combined System

The system has 14 rational solutions of the system above. Some of them are a consequence of others. 11 solutions turned out to be independent:

1) $\{a_2 \to 0, a_3 \to 0, b_2 \to 0\}$;
2) $\{a_2 \to 0, a_3 \to 0, b_3 \to 0\}$;
3) $\{a_1 \to 0, b_1 \to 0, b_2 \to 0\}$;
$\{a_1 \to 0, a_2 \to 0, b_1 \to 0, b_2 \to 0\}$1) is a secuence of 3);
4) $\{a_1 \to 2b_2, a_2 \to 0, b_1 \to 0, b_3 \to 0\}$;
5) $\{a_1 \to 0, a_3 \to 0, b_1 \to 0, b_3 \to 0\}$;
6) $\{a_1 \to 0, b_1 \to 0, b_2 \to 0, b_3 \to 0\}$;
7) $\left\{a_1 \to 0, b_1 \to 0, b_2 \to 0, b_3 \to -\frac{a_2}{2}\right\}$;
$\left\{a_1 \to 0, a_3 \to 0, b_1 \to 0, b_2 \to 0, b_3 \to -\frac{a_2}{2}\right\}$ is a secuence of 7);
8) $\{a_1 \to b_2, a_3 \to 0, b_1 \to 0, b_3 \to a_2\}$;
9) $\{a_1 \to 0, b_1 \to 0, b_2 \to 0, b_3 \to a_2\}$;
10) $\{a_1 \to 0, a_3 \to 0, b_2 \to 0, b_3 \to 2a_2\}$;
11) $\{a_1 \to 0, b_1 \to 0, b_2 \to 0, b_3 \to 2a_2\}$;
$\{a_1 \to 0, a_2 \to 0, b_1 \to 0, b_2 \to 0, b_3 \to 0\}$ is a secuence of 11).

Integrals of the General System

2) $\dot{x} = \alpha x + a_1 x^2, \quad \dot{y} = -y + b_1 x^2 + b_2 xy,$

$$I_2 = x^{1/\alpha}(a_1 x + \alpha)^{-\frac{b_2}{a_1} - \frac{1}{\alpha}} \times \left(b_1 \alpha x \left(\frac{a_1 x}{\alpha} + 1\right)^{\frac{b_2}{a_1} + \frac{1}{\alpha}} {}_2F_1\left(1 + \frac{1}{\alpha}, \frac{b_2}{a_1} + \frac{1}{\alpha}; 2 + \frac{1}{\alpha}; -\frac{a_1 x}{\alpha}\right) - b_1 \alpha x \left(\frac{a_1 x}{\alpha} + 1\right)^{\frac{b_2}{a_1} + \frac{1}{\alpha}} {}_2F_1\left(1 + \frac{1}{\alpha}, \frac{b_2}{a_1} + \frac{1}{\alpha} + 1; 2 + \frac{1}{\alpha}; -\frac{a_1 x}{\alpha}\right) - a_1 \alpha y - a_1 y \right);$$

3) $\dot{x} = \alpha x + a_2 xy + a_3 y^2, \quad \dot{y} = -y + b_3 y^2,$

$$I_3 = y^{\alpha}(1 - b_3 y)^{-\frac{a_2}{b_3} - \alpha} \left(a_3 y^2 (1 - b_3 y)^{\frac{a_2}{b_3} + \alpha} {}_2F_1\left(\alpha + 2, \frac{a_2 + b_3 + b_3 \alpha}{b_3}; \alpha + 3; b_3 y\right) + \alpha x + 2x \right);$$

4) $\dot{x} = \alpha x + 2b_2 x^2 + a_3 y^2, \quad \dot{y} = -y + b_2 xy,$

$$I_4 = y^{2\alpha}\left(a_3 y^2 + (\alpha + 2)x\right)^2 \left(a_3 b_2 y^2 + 2b_2 x + \alpha\right)^{-\alpha - 2};$$

5) $\dot{x} = \alpha x + a_2 xy, \quad \dot{y} = -y + b_2 xy,$

$$I_5 = a_2 y - b_2 x + \log(xy^{\alpha});$$

6) $\dot{x} = \alpha x + a_2 xy + a_3 y^2, \quad \dot{y} = -y,$

$$I_6 = a_2^2 x e^{a_2 y}(-a_2 y)^{\alpha} + a_3 \Gamma(\alpha + 2, -a_2 y);$$

7) $\dot{x} = \alpha x + a_2 xy + a_3 y^2, \quad \dot{y} = -y - 1/2 a_2 y^2,$

$$I_7 = y^{\alpha}(a_2 y + 2)^{-\alpha} \times \left(2a_3 y^2 \left(\tfrac{1}{2} a_2 y + 1\right)^{\alpha} \left(2(\alpha+3)\,{}_2F_1\left(\alpha, \alpha+2; \alpha+3; -\tfrac{1}{2} a_2 y\right) + a_2(\alpha+2) y\,{}_2F_1\left(\alpha, \alpha+3; \alpha+4; -\tfrac{1}{2} a_2 y\right)\right) + (\alpha+2)(\alpha+3) x (a_2 y + 2)^2\right);$$

8) $\dot{x} = \alpha x + b_2 x^2 + a_2 xy, \quad \dot{y} = -y + b_2 xy + a_2 y^2$ by Darboux method:

$$I_8 = xy^{\alpha}(-a_2 \alpha y + b_2 x + \alpha)^{-\alpha-1};$$

9) $\dot{x} = \alpha x + a_2 xy + a_3 y^2, \quad \dot{y} = -y + a_2 y^2;$

$$I_9 = y^{\alpha}(1 - a_2 y)^{-\alpha-1} \times (a_3 y((a_2 y - 1)\,{}_2F_1(1, 1; \alpha+2; a_2 y) + 1) + a_2(\alpha+1)x) y^{\alpha}(1 - a_2 y)^{-\alpha-1}(a_3 y((a_2 y - 1)\,{}_2F_1(1, 1; \alpha+2; a_2 y) + 1) + a_2(\alpha+1)x);$$

10) $\dot{x} = \alpha x + a_2 xy, \quad \dot{y} = -y + b_1 x^2 + 2a_2 y^2,$

$$I_{10} = x^2 \left(-b_1 x^2 + 2\alpha y + y\right)^{2\alpha} \left(-a_2 b_1 x^2 + 2a_2 \alpha y - \alpha\right)^{-2\alpha-1};$$

11) $\dot{x} = \alpha x + a_2 xy + a_3 y^2, \quad \dot{y} = -y + 2a_2 y^2,$

$$I_{11} = y^{\alpha}(1 - 2a_2 y)^{-\alpha-\frac{1}{2}} \left(a_3 y^2 (1 - 2a_2 y)^{\alpha+\frac{1}{2}}\,{}_2F_1\left(\alpha + \tfrac{3}{2}, \alpha+2; \alpha+3; 2a_2 y\right) + \alpha x + 2x\right).$$

At all sets of the parameters which are solutions of the combined algebraic system the general case is integrable. But the first case

$$1) \quad \dot{x} = \alpha x + a_1 x^2, \quad \dot{y} = -y + b_1 x^2 + b_3 y^2$$

has a huge solution. It is integrable case.

Other Examples

We treated the degenerated system [Bruno, Edneral, Romanovski 2017]

$$\begin{cases} \dot{x} = -y^3 - b\,x^3 y + a_0\,x^5 + a_1\,x^2 y^2 \\ \dot{y} = (1/b)\,x^2 y^2 + x^5 + b_0\,x^4 y + b_1\,x\,y^3 \end{cases}$$

with five arbitrary real parameters $b \neq 0, a_1, a_2, b_1, b_2$.

With this technique, we found 7 parameter sets at which the system above is integrable.

For the Liénard-type system

$$\begin{cases} \dot{x} = y \\ \dot{y} = (a_0 + a_1 x)\, y + b_1 x + b_2 x^2 + b_3 x^3 \end{cases}$$

with five arbitrary real parameters a_0, a_1, b_1, b_2, b_3 we we found 6 parameter sets at which the system is integrable [Edneral 2023, in printing].

For the generalized Lunkevich-Sibirskii system

$$\left\{ \begin{array}{l} \dot{x} = \quad y + a_1 x^2 + a_2 xy + a_3 y^2 \\ \dot{y} = -x + b_1 x^2 + b_2 xy + b_3 y^2 \end{array} \right.$$

with six arbitrary real parameters $a_1, a_2, a_3, b_1, b_2, b_3$ we we found 7 parameter sets at which the system is integrable [Edneral, report at the ACA-2022].

Conclusions

- The hypothesis is proposed that local integrability in a neighborhood of each point of some domain of the phase space leads to the existence of the first integral in this domain.
- This hypothesis allows to propose the algorithmic scheme for searching integrable cases.
- For the two-dimensional autonomous polynomial systems we found sets of parameters at which they have the first integrals and, in this way solvable.
- We have the corresponding software.

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Many thanks for your attention