An Algorithm for Solving Two-Sided Linear Vector Equations in Tropical Algebra

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Outline

Max-Plus Algebra

- *Max-plus algebra* $\mathbb{R}_{\max,+}$ is the set of reals with $-\infty$ adjoined
- It is closed under the *addition* $\oplus$ and *multiplication* $\otimes$ defined as
$$x \oplus y = \max(x, y), \qquad x \otimes y = x + y$$
(in what follow the multiplication sign $\otimes$ is omitted to save writing)
- The neutral elements are the zero $\mathbb{0} = -\infty$ and the identity $\mathbb{1} = 0$
- For any $x \neq \mathbb{0}$, the multiplicative inverse x^{-1} is equal to $-x$
- The power x^y corresponds to the arithmetic product yx
- Algebra of matrices and vectors is introduced in the usual way
- Matrix (vector) operations follow standard entrywise formulas with arithmetic addition and multiplication replaced by $\oplus$ and $\otimes$
- For a column vector $\boldsymbol{a} = (a_i)$, its *multiplicative conjugate* is a row vector $\boldsymbol{a}^- = (a_i^-)$ with $a_i^- = a_i^{-1}$ if $a_i \neq \mathbb{0}$, and $a_i^- = \mathbb{0}$ otherwise
- All vectors are considered column vectors unless otherwise stated

Tropical Vector Space

- Consider a system of vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \mathbb{R}^m_{\max,+}$
- A vector $\boldsymbol{b} \in R^m_{\max,+}$ is a *linear combination* of the vectors if

$$\boldsymbol{b} = x_1 \boldsymbol{a}_1 \oplus \cdots \oplus x_n \boldsymbol{a}_n, \qquad x_1, \ldots, x_n \in \mathbb{R}_{\max,+}$$

- The *linear span* defined as the set of all linear combinations

$$\mathcal{A} = \{x_1 \boldsymbol{a}_1 \oplus \cdots \oplus x_n \boldsymbol{a}_n | \ x_1, \ldots, x_n \in \mathbb{R}_{\max,+}\}$$

  is closed under vector addition and scalar multiplication
- The linear span $\mathcal{A}$ is referred to as a *tropical vector (sub)space* that is generated by the system of vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$
- Any vector $\boldsymbol{a} \in \mathcal{A}$ can be represented with the matrix $\boldsymbol{A} = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$ and a vector $\boldsymbol{x} = (x_1, \ldots, x_n)^T$ as the product

$$\boldsymbol{a} = \boldsymbol{A}\boldsymbol{x}$$

Vector Operations and Linear Span in $\mathbb{R}^2_{\max,+}$

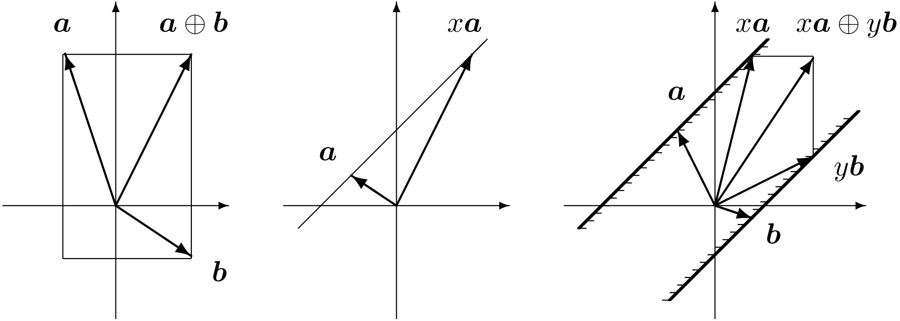


Figure: Addition (left), scalar multiplication (middle), and a linear span (right)

- Addition follows the rectangle law (instead of parallelogram law)
- Multiplication shifts the vector in the direction of 45° to the axes
- Linear span takes the form of a band between lines lying at 45°

Distance Function

- For any vectors $\boldsymbol{a} = (a_i)$ and $\boldsymbol{b} = (b_i)$ without zero components, we define a distance function as follows:

$$d(\boldsymbol{a}, \boldsymbol{b}) = \bigoplus_i \left(b_i^{-1} a_i \oplus a_i^{-1} b_i \right) = \boldsymbol{b}^{-} \boldsymbol{a} \oplus \boldsymbol{a}^{-} \boldsymbol{b}$$

- In the context of max-plus algebra $\mathbb{R}_{\max,+}$, the function d coincides for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^m$ with the Chebyshev metric

$$d_\infty(\boldsymbol{a}, \boldsymbol{b}) = \max_i \max(a_i - b_i, b_i - a_i) = \max_i |b_i - a_i|$$

- Let $\mathcal{A}$ be a vector space generated by a matrix $\boldsymbol{A}$ without zero rows and columns, and $\boldsymbol{b}$ be a vector without zero components
- The distance from the vector $\boldsymbol{b}$ to the vector space $\mathcal{A}$ is given by

$$d(\mathcal{A}, \boldsymbol{b}) = \min_{\boldsymbol{a} \in \mathcal{A}} d(\boldsymbol{a}, \boldsymbol{b}) = \min_{\boldsymbol{x}} d(\boldsymbol{A}\boldsymbol{x}, \boldsymbol{b}) = \min_{\boldsymbol{x}} (\boldsymbol{b}^{-} \boldsymbol{A}\boldsymbol{x} \oplus (\boldsymbol{A}\boldsymbol{x})^{-} \boldsymbol{b})$$

One-Sided Vector Equation

- Let $\mathcal{A}$ be a vector space generated by a matrix $\boldsymbol{A}$ without zero rows and columns, and $\boldsymbol{b}$ be a vector without zero components
- To evaluate the distance between $\boldsymbol{b}$ and $\mathcal{A}$, we denote

$$\Delta = (\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-})^{-}\boldsymbol{b}$$

Lemma (K. 2004,2009)

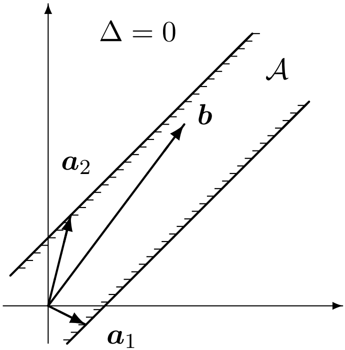
The distance between $\boldsymbol{b}$ and $\mathcal{A}$ is given by

$$\min_{\boldsymbol{x}} d(\boldsymbol{A}\boldsymbol{x}, \boldsymbol{b}) = \sqrt{\Delta},$$

where the minimum is achieved at $\boldsymbol{x} = \sqrt{\Delta}(\boldsymbol{b}^{-}\boldsymbol{A})^{-}$

- The closest vector in $\mathcal{A}$ to the vector $\boldsymbol{b}$ is $\boldsymbol{y} = \sqrt{\Delta}\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-}$
- The value $\Delta = 0$ means that $\boldsymbol{b} \in \mathcal{A}$, while $\Delta > 0$ means $\boldsymbol{b} \notin \mathcal{A}$

Distance From Vector to Tropical Vector Space in $\mathbb{R}^2_{\max,+}$



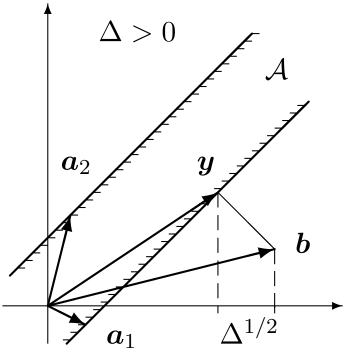


Figure: The vector $\boldsymbol{b}$ is inside the linear span $\mathcal{A}$ (left) and outside $\mathcal{A}$ (right)

- If the condition $\Delta = 0$ holds, then $\boldsymbol{b} \in \mathcal{A} = \{\boldsymbol{A}\boldsymbol{x} | \boldsymbol{x} \in \mathbb{R}^2\}$
- If $\Delta > 0$, then $\boldsymbol{b} \notin \mathcal{A} = \{\boldsymbol{A}\boldsymbol{x} | \boldsymbol{x} \in \mathbb{R}^2\}$
- The Chebyshev distance between $\boldsymbol{b}$ and $\mathcal{A}$ is equal to $\sqrt{\Delta}$

- Let $\boldsymbol{A}$ be a matrix without zero rows and columns, and $\boldsymbol{b}$ a vector without zero components
- Consider the one-sided equation and inequality

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \quad (*), \qquad \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \quad (**)$$

- The next result follows from the above consideration of distances

Theorem (K. 2004,2009)

The following statements hold:

1. *If $\Delta = 0$, then equation $(*)$ has solutions; the vector $\boldsymbol{x} = (\boldsymbol{b}^{-}\boldsymbol{A})^{-}$ is the maximal solution;*
2. *If $\Delta > 0$, then no solution exists; the vector $\boldsymbol{x} = \sqrt{\Delta}(\boldsymbol{b}^{-}\boldsymbol{A})^{-}$ is the best approximate solution*

Lemma

For any matrix $\boldsymbol{A}$ without zero columns and vector $\boldsymbol{b}$ without zero components, all solutions of inequality $(**)$ are given by

$$\boldsymbol{x} \leq (\boldsymbol{b}^{-}\boldsymbol{A})^{-}$$

Solution of One-Sided Inequality $Ax \le b$ in $\mathbb{R}^2_{\max,+}$

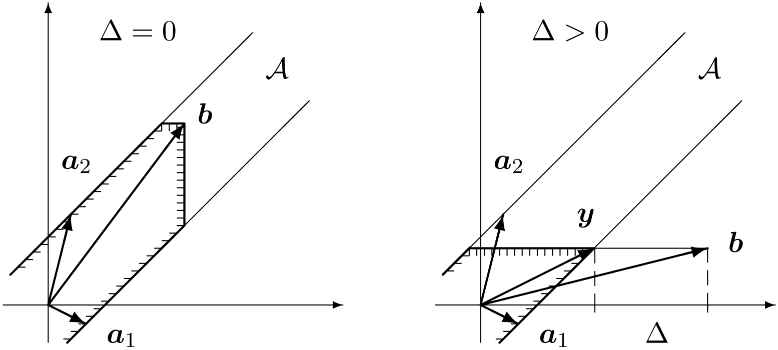


Figure: The vector b is inside the linear span $\mathcal{A}$ (left) and outside $\mathcal{A}$ (right)

- The hatched boundaries indicate the area of vectors Ax corresponding all solution vectors x

Alternating Algorithm

- Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be matrices without zero rows and columns
- Consider the problem to find vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ without zero components to satisfy the two-sided equation

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{B}\boldsymbol{y}$$

Alternating algorithm (Cuninghame-Green & Butkovič, 2003)

- Given a vector $\boldsymbol{x}_0$, the algorithm solves one-sided inequalities

$$\boldsymbol{A}\boldsymbol{x}_0 \geq \boldsymbol{B}\boldsymbol{y}_1, \quad \boldsymbol{A}\boldsymbol{x}_1 \leq \boldsymbol{B}\boldsymbol{y}_1, \quad \boldsymbol{A}\boldsymbol{x}_1 \geq \boldsymbol{B}\boldsymbol{y}_2, \quad \boldsymbol{A}\boldsymbol{x}_2 \leq \boldsymbol{B}\boldsymbol{y}_2, \quad \ldots$$

  to obtain sequences of vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots$ and $\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots$
- Both sequences are known to either converge to some vectors $\boldsymbol{x}_*$ and $\boldsymbol{y}_*$ or properly decrease after a number of steps

Alternating Algorithm in $\mathbb{R}^2_{\max,+}$

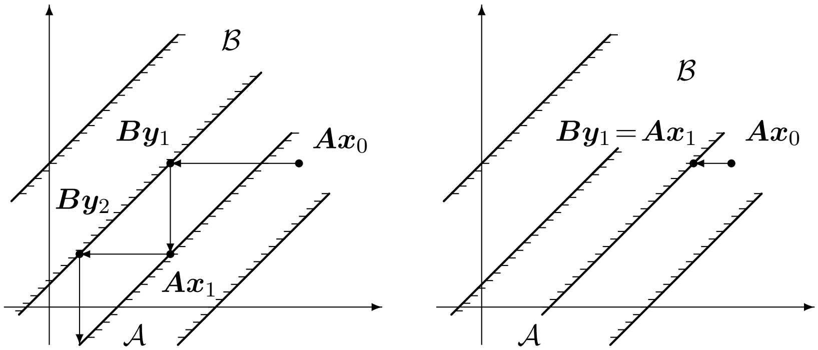


Figure: The sets $\mathcal{A}$ and $\mathcal{B}$ do not intersect (left) and intersect (right)

- The vector $\boldsymbol{y}_1$ is found by solving the inequality $\boldsymbol{B}\boldsymbol{y}_1 \leq \boldsymbol{a}_0 = \boldsymbol{A}\boldsymbol{x}_0$
- The vector $\boldsymbol{x}_1$ is found by solving the inequality $\boldsymbol{A}\boldsymbol{x}_1 \leq \boldsymbol{b}_1 = \boldsymbol{B}\boldsymbol{y}_1$
- ......

Alternating Algorithm

Require: $\boldsymbol{x} > \boldsymbol{0}$

$Stop \leftarrow 0; \quad i \leftarrow 0; \quad \boldsymbol{x}_0 = \boldsymbol{x}$;
while $Stop = 0$ **do**
  $\boldsymbol{y}_{i+1} \leftarrow ((\boldsymbol{A}\boldsymbol{x}_i)^{-}\boldsymbol{B})^{-}$;
  if $\boldsymbol{A}\boldsymbol{x}_i = \boldsymbol{B}\boldsymbol{y}_{i+1}$ or $\boldsymbol{y}_{i+1} < \boldsymbol{y}_i$ **then**
    $\boldsymbol{x}_* \leftarrow \boldsymbol{x}_i; \quad \boldsymbol{y}_* \leftarrow \boldsymbol{y}_{i+1}; \quad Stop \leftarrow 1$;
  else
    $i \leftarrow i + 1$;
  end if
  $\boldsymbol{x}_{i+1} \leftarrow ((\boldsymbol{B}\boldsymbol{y}_i)^{-}\boldsymbol{A})^{-}$;
  if $\boldsymbol{A}\boldsymbol{x}_{i+1} = \boldsymbol{B}\boldsymbol{y}_i$ or $\boldsymbol{x}_{i+1} < \boldsymbol{x}_i$ **then**
    $\boldsymbol{x}_* \leftarrow \boldsymbol{x}_{i+1}; \quad \boldsymbol{y}_* \leftarrow \boldsymbol{y}_i; \quad Stop \leftarrow 1$;
  else
    $i \leftarrow i + 1$;
  end if
end while

Equations Based Alternating

- We propose an algorithm constructing a sequence of vectors taken alternatively from the both tropical vector spaces $\mathcal{A}$ and $\mathcal{B}$
- After selecting a vector in one space, the next vector is found in the other space to minimize the distance to the former vector
- Given a vector $\boldsymbol{x}_0$, the algorithm examines one-sided equations

$$\boldsymbol{A}\boldsymbol{x}_0 = \boldsymbol{B}\boldsymbol{y}_1, \quad \boldsymbol{A}\boldsymbol{x}_1 = \boldsymbol{B}\boldsymbol{y}_1, \quad \boldsymbol{A}\boldsymbol{x}_1 = \boldsymbol{B}\boldsymbol{y}_2, \quad \boldsymbol{A}\boldsymbol{x}_2 = \boldsymbol{B}\boldsymbol{y}_2, \quad \dots$$

- The algorithm successively calculate

$$\boldsymbol{y}_1 = \sqrt{\Delta_0}((\boldsymbol{A}\boldsymbol{x}_0)^{-}\boldsymbol{B})^{-}, \qquad \Delta_0 = (\boldsymbol{B}((\boldsymbol{A}\boldsymbol{x}_0)^{-}\boldsymbol{B})^{-})^{-}\boldsymbol{A}\boldsymbol{x}_0,$$

$$\boldsymbol{x}_2 = \sqrt{\Delta_1}((\boldsymbol{B}\boldsymbol{y}_1)^{-}\boldsymbol{A})^{-}, \qquad \Delta_1 = (\boldsymbol{A}((\boldsymbol{B}\boldsymbol{y}_1)^{-}\boldsymbol{A})^{-})^{-}\boldsymbol{B}\boldsymbol{y}_1, \quad \dots$$

- We show that the sequence $\Delta_0, \Delta_1, \dots$ converges to $\Delta_* \geq 0$
- If $\Delta_* = 0$, then the two-sided equation has a solution $\boldsymbol{x}_*$ and $\boldsymbol{y}_*$
- Otherwise $\boldsymbol{x}_*$ and $\boldsymbol{y}_*$ show one of the best approximate solutions

Equations Based Alternating Algorithm in $\mathbb{R}^2_{\max,+}$

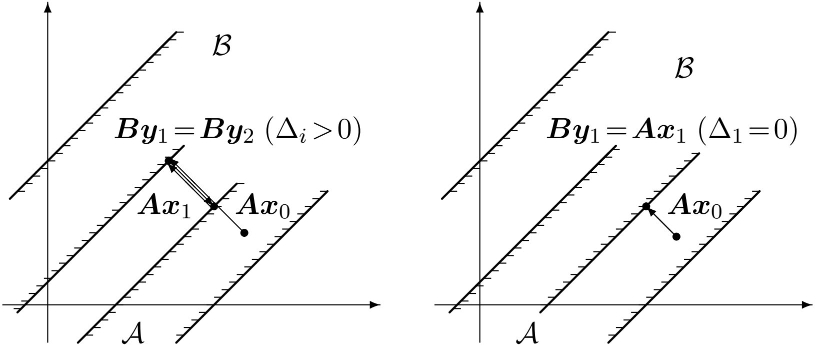


Figure: The sets $\mathcal{A}$ and $\mathcal{B}$ do not intersect (left) and intersect (right)

- The algorithm stops at step i if one of the conditions holds:
  1. $\Delta_i = 0$;
  2. The vector $\boldsymbol{x}_i$ (or $\boldsymbol{y}_i$) coincides with $\boldsymbol{x}_j$ (or $\boldsymbol{y}_j$) for some $j < i$

Equations Based Alternating Algorithm

Require: $\boldsymbol{x} > \boldsymbol{0}$

$Stop \leftarrow 0; \quad i \leftarrow 0; \quad \boldsymbol{x}_0 = \boldsymbol{x}$;
while $Stop = 0$ **do**
  $\Delta_i \leftarrow (\boldsymbol{B}((\boldsymbol{A}\boldsymbol{x}_i)^{-}\boldsymbol{B})^{-})^{-}\boldsymbol{A}\boldsymbol{x}_i; \qquad \boldsymbol{y}_{i+1} \leftarrow \sqrt{\Delta_i}((\boldsymbol{A}\boldsymbol{x}_i)^{-}\boldsymbol{B})^{-}$;
  if $\Delta_i = 0$ or $\boldsymbol{y}_{i+1} = \boldsymbol{y}_j$ for some $j < i$ **then**
    $\Delta_* \leftarrow \Delta_i; \quad \boldsymbol{x}_* \leftarrow \boldsymbol{x}_i; \quad \boldsymbol{y}_* \leftarrow \boldsymbol{y}_{i+1}; \quad Stop \leftarrow 1$;
  else
    $i \leftarrow i + 1$;
  end if
  $\Delta_i \leftarrow (\boldsymbol{A}((\boldsymbol{B}\boldsymbol{y}_i)^{-}\boldsymbol{A})^{-})^{-}\boldsymbol{B}\boldsymbol{y}_i; \qquad \boldsymbol{x}_{i+1} \leftarrow \sqrt{\Delta_i}((\boldsymbol{B}\boldsymbol{y}_i)^{-}\boldsymbol{A})^{-}$;
  if $\Delta_i = 0$ or $\boldsymbol{x}_{i+1} = \boldsymbol{x}_j$ for some $j < i$ **then**
    $\Delta_* \leftarrow \Delta_i; \quad \boldsymbol{x}_* \leftarrow \boldsymbol{x}_{i+1}; \quad \boldsymbol{y}_* \leftarrow \boldsymbol{y}_i; \quad Stop \leftarrow 1$;
  else
    $i \leftarrow i + 1$;
  end if
end while

Conclusion

- We have proposed a solution procedure based on alternatively solving one-sided equations rather than one-sided inequalities
- The procedure admits a clear geometrical explanation in terms of distances between vectors in tropical vector spaces
- The deviation between both sides of equation are evaluated for iterations, which can be useful in finding approximate solutions