Implementation of A.N.Krylov series convegence acceleration in the CAS Sage

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Ksaverii Malyshev, Mikhail Malykh
(SINP MSU, RUDN)

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Problem formulation

We investigate the application of computer algebra systems to the summation of trigonometric Fourier series. Fourier series associated with problems of mathematical physics are not analytic functions of a complex argument. Therefore, an attempt to find the sum of the Fourier series in closed form using CAS leads to transcendental functions. At the same time, often these sums are elementary functions of a real variable, piecewise given elementary functions. This class of functions is not included in the class of elementary Liouville functions.

In this talk, we present the first functions of the «Kryloff for Sage» software package, which make it possible to determine at least some of the cases in which the Fourier series represents an elementary function of a real argument.

Start point: simplest Green's function

$$\begin{cases} \dfrac{\partial^2 g}{\partial t^2} = \dfrac{\partial^2 g}{\partial x^2}, 0 < x < \pi, t > \tau, 0 \leq \tau < +\infty; \\ g|_{t=\tau} = 0,\ \dfrac{\partial g}{\partial t}\Big|_{t=\tau} = \delta(x-s), 0 < x < \pi, 0 < s < \pi; \\ g|_{x=0} = 0,\ g|_{x=l} = 0,\ \tau < t < +\infty \end{cases} \quad (1)$$

Here $\delta(x-s)$ is Dirac delta function. The Green's function g can be found as a Fourier series [Tikhonov, Samarskii]:

$$g = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n(t-\tau)\sin ns \sin nx. \quad (2)$$

This series has been studied by many authors, but is not presented in the available literature in finite terms.

Start point: simplest Green's function

Theorem

The Green's function g can be presented in finite terms as the expression:

$$g = \frac{1}{2\pi}\left(\operatorname{atan}\left(\cot\frac{t-\tau-x+s}{2}\right) + \operatorname{atan}\left(\cot\frac{t-\tau+x-s}{2}\right) - \right.$$
$$\left. - \operatorname{atan}\left(\cot\frac{t-\tau-x-s}{2}\right) - \operatorname{atan}\left(\cot\frac{x+s+t-\tau}{2}\right)\right)$$

Alternative expression has the form:

$$g = \frac{1}{2}\left(\left[\frac{x-s+T}{2\pi}\right] + \left[\frac{-x+s+T}{2\pi}\right] \right.$$
$$\left. - \left[\frac{-x-s+T}{2\pi}\right] - \left[\frac{x+s+T}{2\pi}\right]\right).$$

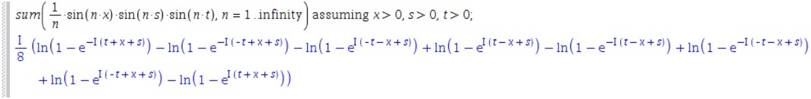
Here $[\cdot]$ means «floor», $T = t - \tau$.

Our Research

Leaving aside further mathematical studies of Green's functions, we fix the following questions:

- Can we find such kind of expressions in finite terms of Fourier series in moderm CAS?
- What kind or expressions for series can we find for Green's functions and other Fourier series using computer instruments?
- Is it possible to systematize the cases in which the sum of the series presents in finite terms? Does such a systematization allow implementation in CAS?

Fourier series summation, g

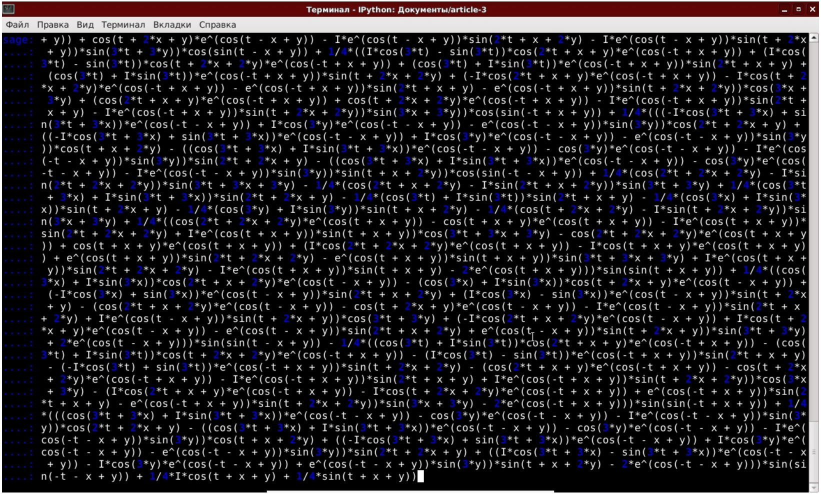


$$sum\left(\frac{1}{n}\cdot\sin(n\cdot x)\cdot\sin(n\cdot s)\cdot\sin(n\cdot t),\ n=1..\text{infinity}\right)\text{ assuming } x>0,\ s>0,\ t>0;$$

$$\frac{I}{8}\left(\ln\left(1-e^{-I(t+x+s)}\right)-\ln\left(1-e^{-I(-t+x+s)}\right)-\ln\left(1-e^{I(-t-x+s)}\right)+\ln\left(1-e^{I(t-x+s)}\right)-\ln\left(1-e^{-I(t-x+s)}\right)+\ln\left(1-e^{-I(-t-x+s)}\right)\right.$$
$$\left.+\ln\left(1-e^{I(-t+x+s)}\right)-\ln\left(1-e^{I(t+x+s)}\right)\right)$$

The simplest Green's function g in finite terms in CAS Maple'2019. We can see satisfactory result, but too difficult for users. We cannot find this result without assuming, because this representation does not hold for arbitrary complex values of x, s, t.

Fourier series summation, g



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sage: + y)) + cos(t + 2*x + y)*e^(cos(t - x + y)) - I*e^(cos(t - x + y))*sin(2*t + x + 2*y) - I*e^(cos(t - x + y))*sin(t + 2*x
....:  + y))*sin(3*t + 3*y))*cos(sin(t - x + y)) + 1/4*((I*cos(3*t) - sin(3*t))*cos(2*t + x + y)*e^(cos(-t + x + y)) + (I*cos(
....: 3*t) - sin(3*t))*cos(t + 2*x + 2*y)*e^(cos(-t + x + y)) + (cos(3*t) + I*sin(3*t))*e^(cos(-t + x + y))*sin(2*t + x + y) +
....:  (cos(3*t) + I*sin(3*t))*e^(cos(-t + x + y))*sin(t + 2*x + 2*y) + (-I*cos(2*t + x + y)*e^(cos(-t + x + y)) - I*cos(t + 2
....: *x + 2*y)*e^(cos(-t + x + y)) - e^(cos(-t + x + y))*sin(2*t + x + y) - e^(cos(-t + x + y))*sin(t + 2*x + 2*y))*cos(3*x +
....:  3*y) + (cos(2*t + x + y)*e^(cos(-t + x + y)) + cos(t + 2*x + 2*y)*e^(cos(-t + x + y)) - I*e^(cos(-t + x + y))*sin(2*t +
....:  x + y) - I*e^(cos(-t + x + y))*sin(t + 2*x + 2*y))*sin(3*x + 3*y))*cos(sin(-t + x + y)) + 1/4*(((-I*cos(3*t + 3*x) + si
....: n(3*t + 3*x))*e^(cos(-t - x + y)) + I*cos(3*y)*e^(cos(-t - x + y)) - e^(cos(-t - x + y))*sin(3*y))*cos(2*t + 2*x + y) +
....: ((-I*cos(3*t + 3*x) + sin(3*t + 3*x))*e^(cos(-t - x + y)) + I*cos(3*y)*e^(cos(-t - x + y)) - e^(cos(-t - x + y))*sin(3*y
....: ))*cos(t + x + 2*y) - ((cos(3*t + 3*x) + I*sin(3*t + 3*x))*e^(cos(-t - x + y)) - cos(3*y)*e^(cos(-t - x + y)) - I*e^(cos
....: (-t - x + y))*sin(3*y))*sin(2*t + 2*x + y) - ((cos(3*t + 3*x) + I*sin(3*t + 3*x))*e^(cos(-t - x + y)) - cos(3*y)*e^(cos(
....: -t - x + y)) - I*e^(cos(-t - x + y))*sin(3*y))*sin(t + x + 2*y))*cos(sin(-t - x + y)) + 1/4*(cos(2*t + 2*x + 2*y) - I*si
....: n(2*t + 2*x + 2*y))*sin(3*t + 3*x + 3*y) - 1/4*(cos(2*t + x + 2*y) - I*sin(2*t + x + 2*y))*sin(3*t + 3*y) + 1/4*(cos(3*t
....:  + 3*x) + I*sin(3*t + 3*x))*sin(2*t + 2*x + y) - 1/4*(cos(3*t) + I*sin(3*t))*sin(2*t + x + y) - 1/4*(cos(3*x) + I*sin(3*
....: x))*sin(t + 2*x + y) - 1/4*(cos(3*y) + I*sin(3*y))*sin(t + x + 2*y) - 1/4*(cos(t + 2*x + 2*y) - I*sin(t + 2*x + 2*y))*si
....: n(3*x + 3*y) + 1/4*((cos(2*t + 2*x + 2*y)*e^(cos(t + x + y)) - cos(t + x + y)*e^(cos(t + x + y)) - I*e^(cos(t + x + y))*
....: sin(2*t + 2*x + 2*y) + I*e^(cos(t + x + y))*sin(t + x + y))*cos(3*t + 3*x + 3*y) - cos(2*t + 2*x + 2*y)*e^(cos(t + x + y
....: )) + cos(t + x + y)*e^(cos(t + x + y)) + (I*cos(2*t + 2*x + 2*y)*e^(cos(t + x + y)) - I*cos(t + x + y)*e^(cos(t + x + y)
....: ) + e^(cos(t + x + y))*sin(2*t + 2*x + 2*y) - e^(cos(t + x + y))*sin(t + x + y))*sin(3*t + 3*x + 3*y) + I*e^(cos(t + x +
....:  y))*sin(2*t + 2*x + 2*y) - I*e^(cos(t + x + y))*sin(t + x + y) - 2*e^(cos(t + x + y)))*sin(sin(t + x + y)) + 1/4*((cos(
....: 3*x) + I*sin(3*x))*cos(2*t + x + 2*y)*e^(cos(t - x + y)) - (cos(3*x) + I*sin(3*x))*cos(t + 2*x + y)*e^(cos(t - x + y)) +
....:  (-I*cos(3*x) + sin(3*x))*e^(cos(t - x + y))*sin(2*t + x + 2*y) + (I*cos(3*x) - sin(3*x))*e^(cos(t - x + y))*sin(t + 2*x
....:  + y) - (cos(2*t + x + 2*y)*e^(cos(t - x + y)) - cos(t + 2*x + y)*e^(cos(t - x + y)) - I*e^(cos(t - x + y))*sin(2*t + x
....: + 2*y) + I*e^(cos(t - x + y))*sin(t + 2*x + y))*cos(3*t + 3*y) + (-I*cos(2*t + x + 2*y)*e^(cos(t - x + y)) + I*cos(t + 2
....: *x + y)*e^(cos(t - x + y)) - e^(cos(t - x + y))*sin(2*t + x + 2*y) + e^(cos(t - x + y))*sin(t + 2*x + y))*sin(3*t + 3*y)
....:  + 2*e^(cos(t - x + y)))*sin(sin(t - x + y)) - 1/4*((cos(3*t) + I*sin(3*t))*cos(2*t + x + y)*e^(cos(-t + x + y)) - (cos(
....: 3*t) + I*sin(3*t))*cos(t + 2*x + 2*y)*e^(cos(-t + x + y)) - (I*cos(3*t) - sin(3*t))*e^(cos(-t + x + y))*sin(2*t + x + y)
....:  - (-I*cos(3*t) + sin(3*t))*e^(cos(-t + x + y))*sin(t + 2*x + 2*y) - (cos(2*t + x + y)*e^(cos(-t + x + y)) - cos(t + 2*x
....:  + 2*y)*e^(cos(-t + x + y)) - I*e^(cos(-t + x + y))*sin(2*t + x + y) + I*e^(cos(-t + x + y))*sin(t + 2*x + 2*y))*cos(3*x
....:  + 3*y) - (I*cos(2*t + x + y)*e^(cos(-t + x + y)) - I*cos(t + 2*x + 2*y)*e^(cos(-t + x + y)) + e^(cos(-t + x + y))*sin(2
....: *t + x + y) - e^(cos(-t + x + y))*sin(t + 2*x + 2*y))*sin(3*x + 3*y) - 2*e^(cos(-t + x + y)))*sin(sin(-t + x + y)) + 1/4
....: *(((cos(3*t + 3*x) + I*sin(3*t + 3*x))*e^(cos(-t - x + y)) - cos(3*y)*e^(cos(-t - x + y)) - I*e^(cos(-t - x + y))*sin(3*
....: y))*cos(2*t + 2*x + y) - ((cos(3*t + 3*x) + I*sin(3*t + 3*x))*e^(cos(-t - x + y)) - cos(3*y)*e^(cos(-t - x + y)) - I*e^(
....: cos(-t - x + y))*sin(3*y))*cos(t + x + 2*y) + ((-I*cos(3*t + 3*x) + sin(3*t + 3*x))*e^(cos(-t - x + y)) + I*cos(3*y)*e^(
....: cos(-t - x + y)) - e^(cos(-t - x + y))*sin(3*y))*sin(2*t + 2*x + y) + ((I*cos(3*t + 3*x) - sin(3*t + 3*x))*e^(cos(-t - x
....:  + y)) - I*cos(3*y)*e^(cos(-t - x + y)) + e^(cos(-t - x + y))*sin(3*y))*sin(t + x + 2*y) - 2*e^(cos(-t - x + y)))*sin(si
....: n(-t - x + y)) + 1/4*I*cos(t + x + y) + 1/4*sin(t + x + y))
```

The simplest Green's function g in finite terms in CAS Sage.

Another Green's function, $\tilde{g}$

Consider the series:

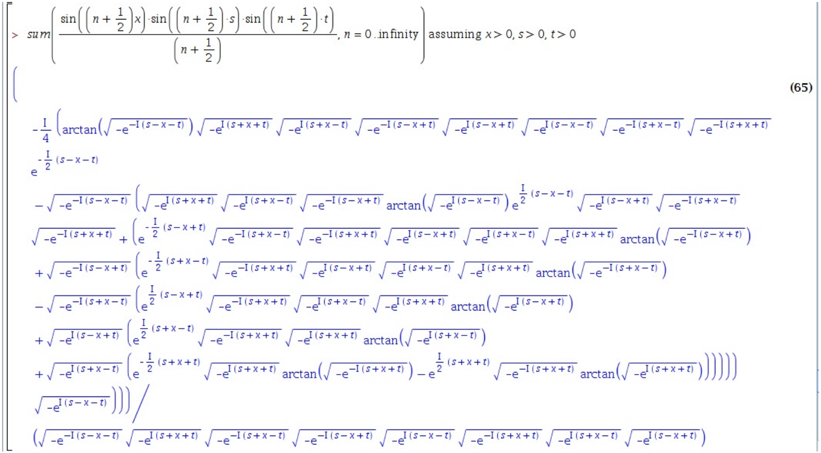
$$\tilde{g} = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin\left(k+\frac{1}{2}\right) x \sin\left(k+\frac{1}{2}\right) s \sin\left(k+\frac{1}{2}\right)(t-\tau).$$

In finite terms:

$$\tilde{g} = \frac{1}{4}\Bigg(\operatorname{sign}\sin\frac{x-s+t-\tau}{2} + \operatorname{sign}\sin\frac{-x+s+t-\tau}{2} + \\ \operatorname{sign}\sin\frac{x+s-t+\tau}{2} + \operatorname{sign}\sin\frac{-x-s-t+\tau}{2}\Bigg).$$

This test turned out to be much more difficult.

Another Green's function, $\tilde{g}$



$$> sum\left(\frac{\sin\left(\left(n+\frac{1}{2}\right)x\right)\cdot\sin\left(\left(n+\frac{1}{2}\right)\cdot s\right)\cdot\sin\left(\left(n+\frac{1}{2}\right)\cdot t\right)}{\left(n+\frac{1}{2}\right)}, n=0\,..\,\mathrm{infinity}\right)\ \mathrm{assuming}\ x>0,\ s>0,\ t>0$$

$$\begin{gathered}
(\qquad\qquad (65) \\
-\frac{I}{4}\Big(\arctan\left(\sqrt{-e^{-I(s-x-t)}}\right)\sqrt{-e^{I(s+x+t)}}\,\sqrt{-e^{I(s+x-t)}}\,\sqrt{-e^{-I(s-x+t)}}\,\sqrt{-e^{I(s-x+t)}}\,\sqrt{-e^{I(s-x-t)}}\,\sqrt{-e^{-I(s+x-t)}}\,\sqrt{-e^{-I(s+x+t)}} \\
e^{-\frac{I}{2}(s-x-t)} \\
-\sqrt{-e^{-I(s-x-t)}}\Big(\sqrt{-e^{I(s+x+t)}}\,\sqrt{-e^{I(s+x-t)}}\,\sqrt{-e^{-I(s-x+t)}}\,\arctan\left(\sqrt{-e^{I(s-x-t)}}\right)e^{\frac{I}{2}(s-x-t)}\sqrt{-e^{I(s-x+t)}}\,\sqrt{-e^{-I(s+x-t)}} \\
\sqrt{-e^{-I(s+x+t)}}+\Big(e^{-\frac{I}{2}(s-x+t)}\sqrt{-e^{-I(s+x-t)}}\,\sqrt{-e^{-I(s+x+t)}}\,\sqrt{-e^{I(s-x+t)}}\,\sqrt{-e^{I(s+x-t)}}\,\sqrt{-e^{I(s+x+t)}}\,\arctan\left(\sqrt{-e^{-I(s-x+t)}}\right) \\
+\sqrt{-e^{-I(s-x+t)}}\Big(e^{-\frac{I}{2}(s+x-t)}\sqrt{-e^{-I(s+x+t)}}\,\sqrt{-e^{I(s-x+t)}}\,\sqrt{-e^{I(s+x-t)}}\,\sqrt{-e^{I(s+x+t)}}\,\arctan\left(\sqrt{-e^{-I(s+x-t)}}\right) \\
-\sqrt{-e^{-I(s+x-t)}}\Big(e^{\frac{I}{2}(s-x+t)}\sqrt{-e^{-I(s+x+t)}}\,\sqrt{-e^{I(s+x-t)}}\,\sqrt{-e^{I(s+x+t)}}\,\arctan\left(\sqrt{-e^{I(s-x+t)}}\right) \\
+\sqrt{-e^{I(s-x+t)}}\Big(e^{\frac{I}{2}(s+x-t)}\sqrt{-e^{-I(s+x+t)}}\,\sqrt{-e^{I(s+x+t)}}\,\arctan\left(\sqrt{-e^{I(s+x-t)}}\right) \\
+\sqrt{-e^{I(s+x-t)}}\Big(e^{-\frac{I}{2}(s+x+t)}\sqrt{-e^{I(s+x+t)}}\,\arctan\left(\sqrt{-e^{-I(s+x+t)}}\right)-e^{\frac{I}{2}(s+x+t)}\sqrt{-e^{-I(s+x+t)}}\,\arctan\left(\sqrt{-e^{I(s+x+t)}}\right)\Big)\Big)\Big)\Big)\Big) \\
\sqrt{-e^{I(s-x-t)}}\Big)\Big)\Big)\Big/ \\
\left(\sqrt{-e^{-I(s-x-t)}}\,\sqrt{-e^{I(s+x+t)}}\,\sqrt{-e^{-I(s+x-t)}}\,\sqrt{-e^{-I(s-x+t)}}\,\sqrt{-e^{I(s-x-t)}}\,\sqrt{-e^{-I(s+x+t)}}\,\sqrt{-e^{I(s+x-t)}}\,\sqrt{-e^{I(s-x+t)}}\right)
\end{gathered}$$

Fourier testing results: tipycal series from textbook

This is the series for motion of finite string. It satisfies the initial condition $u(0,x) = \varphi(x) = x^2(1-x)$.

$$u = \sum_{n=1}^{\infty} \frac{8 \cdot (-1)^{n+1} - 4}{\pi^3 n^3} \sin(\pi n x) \cos(\pi n c t)$$

Maple'2019 is able to convert the infinite series in symbolic expression, namely

$$\begin{aligned} u = \frac{2i}{\pi^3} \Big(& \mathrm{Li}_3(-e^{i\cdot\pi(x+ct)}) - \mathrm{Li}_3(-e^{-i\cdot\pi(x+ct)}) + \\ & + \mathrm{Li}_3(-e^{i\cdot\pi(x-ct)}) - \mathrm{Li}_3(-e^{-i\cdot\pi(x-ct)}) \Big) - \\ - \frac{i}{\pi^3} \Big(& \mathrm{Li}_3(e^{-i\cdot\pi(x+ct)}) - \mathrm{Li}_3(e^{i\cdot\pi(x+ct)}) + \\ & + \mathrm{Li}_3(e^{-i\cdot\pi(x-ct)}) - \mathrm{Li}_3(e^{i\cdot\pi(x-ct)}) \Big) . \end{aligned}$$

Here $\mathrm{Li}_3(z)$ is Euler's polylogarithm.

Fourier testing results: tipycal series from textbook

But $obviously$, at every moment of time it is a piecewise polynomial function! It is obviously, because we can convert the product $sin \cdot cos$ to the sum of sines. The symbolic expression of the function u in finite terms again requires the piecewise constructions sign or $\arctan(\cot)$. There is an alternative: work in the field of complex numbers $\mathbb{C}$ and use special functions, or work in the field of real numbers $\mathbb{R}$ and use piecewise elementary functions. The bridge between the two representations in finite terms is the Fourier series.

A.N. Krylov's technique Fourier series

The direct application of CAS to the summation of Fourier series can lead to difficulties.

One can try to overcome them by changing the formulation of the problem: instead of the summation problem in the finite terms, consider the problem of accelerating the convergence of the Fourier series. As Krylov wrote, this technique «often leads to the representation of the sum of the proposed series in closed form under the guise piecewice function».

Krylov’s technique in moderm mathemathical physics

- А. Б. Нерсесян “Ускорение сходимости разложений по собственным функциям”, Докл. НАН Армении. 2007.
- А. П. Хромов, М. Ш. Бурлуцкая, “Классическое решение методом Фурье смешанных задач при минимальных требованиях на исходные данные”, Изв. Сарат. ун-та. Нов. сер. Сер. Математика. Механика. Информатика, 14:2 (2014), 171–198
- Adcock B. Modified Fourier expansions: theory, construction and application / Trinity Hall University of Cambridge, 2010.

A.N. Krylov's method [L.V.Kantorovich, V.I. Krylov]

$$\sum_{i=1}^{\infty}\left(U\left(\frac{1}{n}\right)\sin nx + V\left(\frac{1}{n}\right)\cos nx\right)$$

$$U\left(\frac{1}{n}\right) = u_1\frac{1}{n} + u_2\frac{1}{n^2} + O\left(\frac{1}{n^3}\right), u_1, u_2 \in \mathbb{R};$$

$$V\left(\frac{1}{n}\right) = v_1\frac{1}{n} + v_2\frac{1}{n^2} + O\left(\frac{1}{n^3}\right), v_1, v_2 \in \mathbb{R};$$

$$\sum_{n=1}^{\infty}\frac{\sin nx}{n} = \frac{\pi - x}{2}, x \in (0, 2\pi]$$

$$\sum_{n=1}^{\infty}\frac{\cos nx}{n} = -\log 2\left|\sin\frac{x}{2}\right|, x \in (0, 2\pi)$$

We can see, that the simplest Green's function g is the subject, when this scheme has only ONE nontrivial step.

Implementation in Sage

We will implement the above convergence acceleration scheme, assuming that the given functions U and V are good enough. In practice, the Fourier coefficient is a function of n, and it may not be possible to expand it into a series in terms of $\frac{1}{n}$. Let's take a class of functions where such a possibility exists: rational functions of n. The main direction of our work is the symbolic study of a series, and this class of functions has additional advantages:

- The built-in functions of Sage allow to determine the membership of a polynomial ring and its field of quotients. This allows, sometimes, to determine: is a given Fourier series a (piecewise) elementary function?
- The fact that the series diverges can be easily established.
- For this case, standard symbolic tools give the answer in the form of transcendental functions, and comparison of the results may be of independent interest.

Implementation in Sage

For simplicity, today we will leave aside the question of trigonometric transformations and the question of the periodic extension of a piecewise function to a straight line.

Main functions of the program

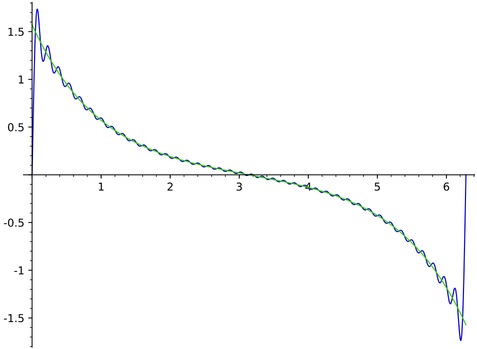
Here U is the function $U(n)$ given by symbolic expression. The prefixes c and s are used to indicate which series is being considered: by cosines (c) or by sines (s).

- is_elementary_c(V)|
- c_series(V,M)|, summation_c(V)|. Finding a partial sum of order M and a sum in closed form using sage tools.
- kryloff_c_slow(V,k)| Returns the expression in closed form for the slowly converging part of the Fourier series. The Taylor polynomial of order k is used. Only those terms are singled out that lead to expressions in elementary functions (logarithm and Bernoulli polynoms).
- c_rapid(V,k,M)| Returns the order M partial sum of the accelerated convergence series.

Examples

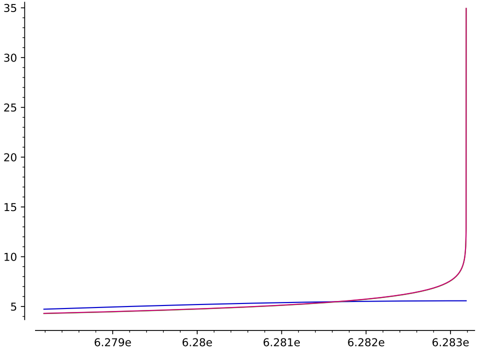
```
sage: load('kryloff-1.sage')\\
sage: U=x/(x^2+1)\\
sage: summation_s(U)\\
1/4*(imag_part(hypergeometric((2, -I + 1, I + 1), (-I + 2, I
sage: f=s_series(U,40)\\
sage: g=kryloff_s_slow(U,5)+s_rapid(U,5,3)\\
sage: m=plot([f,g],z,0,2*pi)
```

Examples



The series $\sum_{n=1}^{\infty} \frac{n}{n^2+1} \sin nx$. Blue line: partial sum of 40 terms. Green line: the sum of elementary part of 5 terms and 3 terms accelerated series.

Examples: logarithm singularity



The series $\sum_{n=1}^{\infty} \frac{1}{n+1} \cos nx$.

Blue line: partial sum of 400 terms. Green line: the sum of elementary part of 5 terms and 5 terms accelerated series. Red line: elementary answer (but this is not basic series).

Summary: results, presented in the talk

1. We present a simple emplementation of A.N. Krylov's metod of Fourier series convergence acceleration
2. This emlementation can give closed-form representation for several cases of Fourier series
3. Functions of the package «Kryloff for Sage» can be adapted to another eigenfunctions: $\sin(n+\frac{1}{2})$ and other cases of series, typical for mathematical physics

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Thank you for your time and attention!