COMPUTING THE COMPLEMENT TO THE AMOEBA OF A MULTIVARIATE POLYNOMIAL

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DEFINITION. **The Reinhardt diagram of a domain $D \subset \mathbb{C}^n$ is the image of D under the map**

$$(x_1, \ldots, x_n) \longmapsto (|x_1|, \ldots, |x_n|).$$

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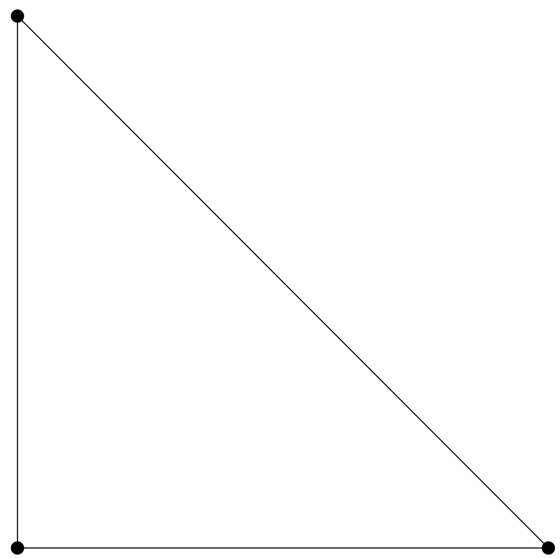
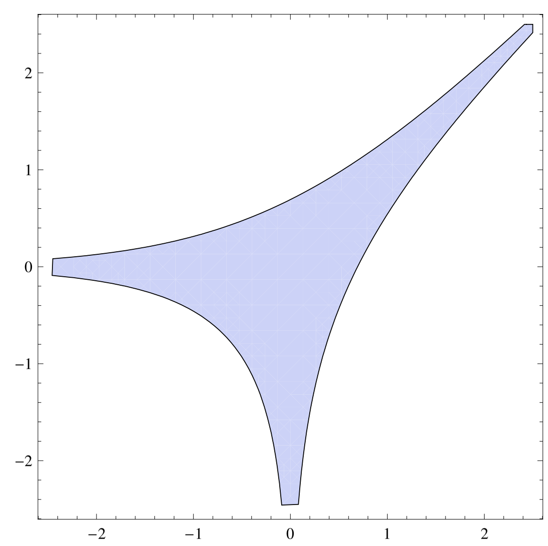
DEFINITION. The amoeba $\mathcal{A}_f$ of a Laurent polynomial $f(x)$ (or of the algebraic hypersurface $\{p(x) = 0\}$) is defined to be the image of the hypersurface $p^{-1}(0)$ under the map

$$\mathrm{Log} : (x_1, \ldots, x_n) \mapsto (\log|x_1|, \ldots, \log|x_n|).$$

EXAMPLE. The amoeba of a complex line.

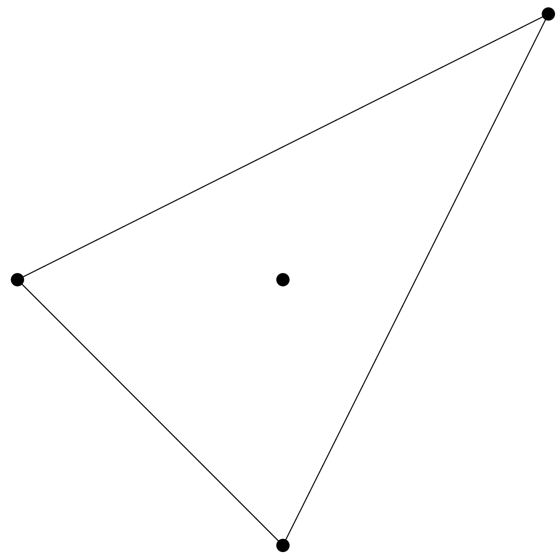
$$p(x, y) = 1 + x + y$$

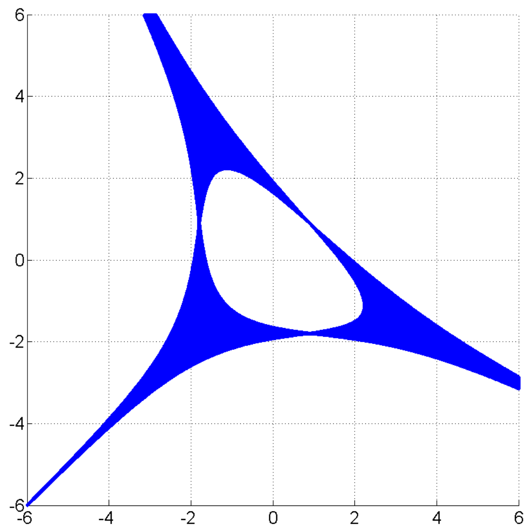




EXAMPLE.

$$p(x, y) = x + y + 6xy + x^2y^2$$





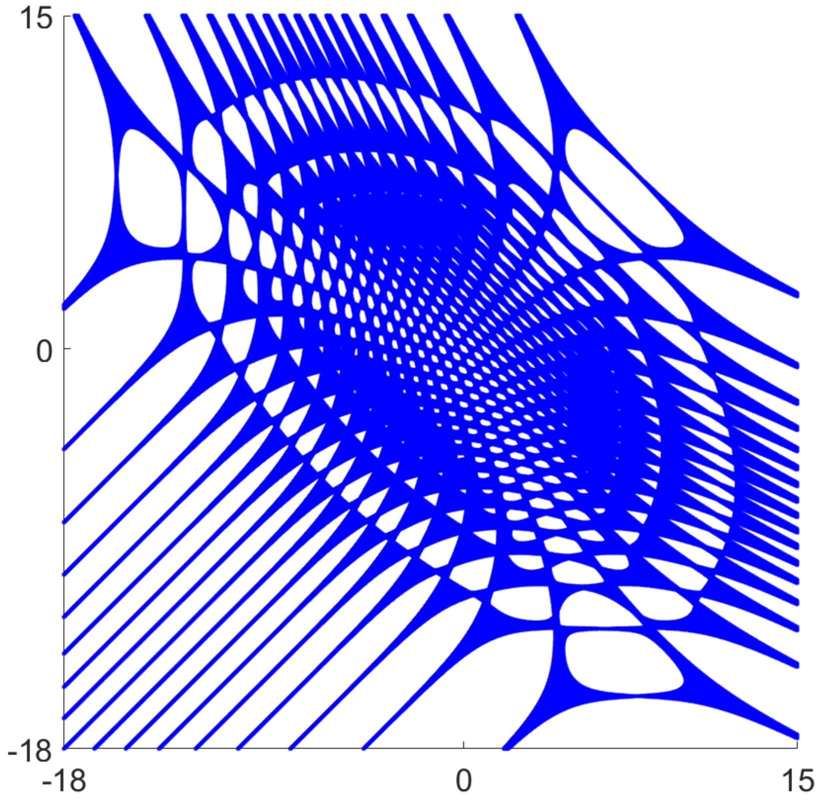


Fig. 1. The amoeba of a hypergeometric polynomial

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$$\{(s, t) \in \mathbb{R}^2 : 1 \leqslant e^s + e^t,\ e^s - 1 \leqslant e^t \leqslant e^s + 1\}.$$

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- The amoeba of the polynomial $p(x, y) = a + x + y + xy$ with an arbitrary complex parameter a is the set of solutions to the inequality

$$\left(e^{2s} - |a|^2\right)^2 + \left(e^{2s} - 1\right)^2 e^{4t} -$$
$$2e^{2t}\left(e^{2s}\left(|a|^2 - 4\,\mathrm{Re}\,a + e^{2s} + 1\right) + |a|^2\right) \leqslant 0.$$

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• The complexity of the boundary of the amoeba of a polynomial is rapidly increasing with its degree and the number of variables. Nevertheless, the contours of the amoebas of classical and generalized discriminants admits the birational Horn–Kapranov parameterization.

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There exists an injective function

$$\nu : \{M\} \longrightarrow \mathbb{Z}^n \cap \mathcal{N}_{p(x)}$$

such that the cone which is dual to $\mathcal{N}_{p(x)}$ at the point $\nu(M)$ coincides with the recession cone of M.

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DEFINITION. If the lower bound is attained in the above theorem then $\mathcal{A}_{p(x)}$ is said to be solid.

1. What is the range of the map $\nu : \{M\} \longrightarrow \mathbb{Z}^n \cap \mathcal{N}_{p(x)}$?

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1. **What is the range of the map $\nu : \{M\} \longrightarrow \mathbb{Z}^n \cap \mathcal{N}_{p(x)}$?**
2. **Where in $\mathbb{R}^n$ is the amoeba $\mathcal{A}_{p(x)}$ located?**

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2. Where in $\mathbb{R}^n$ is the amoeba $\mathcal{A}_{p(x)}$ located?

3. For a given order $v \in \mathbb{Z}^n \cap \mathcal{N}_{p(x)}$, find a point $x \in \mathbb{R}^n$ which belongs to the connected component of $M \subset^c \mathcal{A}_{p(x)}$ with order v.

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1. Except the trivial one-dimensional case, the amoeba of a polynomial is an unbounded subset of the real linear space.

2. The amoeba $\mathcal{A}_f$ of a bivariate polynomial f has finite area. In three and higher dimensions,

$$\frac{\mathrm{vol}(\mathcal{A}_f \cap B(a,r))}{\mathrm{vol}(B(a,r))} \to 0, \quad r \to 0$$

for any polynomial f and any point $a \in \mathbb{R}^n$. So, determining the position of the amoeba in $\mathbb{R}^n$ is in itself a nontrivial problem.

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4. To compute the amoeba of an e.g. bivariate tenth-degree polynomial in the square $[-20, 20] \times [-20, 20]$ we need to operate with values on the order of $e^{10\cdot 20} \simeq 7.22597 \times 10^{86}$, maintaining the computational accuracy of the scale $e^{-10\cdot 20} \simeq 1.38389 \times 10^{-87}$.

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Here by a maximally sparse polynomial we mean a polynomial whose support equals the set of vertices of its Newton polytope.

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I.M. Gelfand, M.M. Kapranov, and A.V. Zelevinsky. Discriminants, resultants and multidimensional determinants. Birkhäuser, 1994.

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