Algorithm for constructing a vector continued fraction

Alexander Kotochigov

Abstract. The talk is devoted to the description of a numerical algorithm - a direct analogue of the expansion of a number into a continued fraction, for vectors. The algorithm solves the problem: constructing for a given numerical vector a sequence of integer vectors such that from them it is possible to restore a sequence of vectors with rational components, approximating the original vector.

Euclid's algorithm allows for any real number α to form a sequence of natural numbers $a_1, a_2, \ldots$ that can be used to form a continued fraction

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}, \quad \frac{p_n}{q_n} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots \cfrac{1}{a_n}}}, \quad \lim_n \frac{p_n}{q_n} = \alpha$$

interest in such cumbersome structures has always been stimulated by the fact that the numbers $\frac{p_n}{q_n}$ – convergents, turns out to be the best rational approximations of α by rational numbers, with denominators not exceeding q_n. Geometrically, this means that in a rectangle with diagonal $(0,0)$, (q_n, p_n) the point (q_n, p_n) is the point of the integer lattice closest to the line $y = \alpha x$, except for the origin.

The problem of finding points of an integer lattice in R^n closest to a given line

$$y_k = \alpha_k t, \ k = 1, \ldots, n$$

obviously remains, but the algorithm for finding such points disappears.

Euler was also interested in stopping the question in this way, and in the works of Jacobi, Voronoi [1], such studies were continued. These tasks are actively considered and are now in the works Bruno A.S.[2], Kokorina [3], Herman O.N.[4] geometric properties are investigated the set of points of the integer lattice closest to the given line. In the book [5] collected a unique bibliography of works somehow related to continued fractions.

This message is devoted to the description of a numerical algorithm - a direct analogue of the expansion of a number into a continued fraction, for vectors (*vector continued fraction*). Decomposition algorithm allows understanding of convergent fractions in the same sense as for ordinary continued fractions.

The algorithm solves the problem: constructing for a given numerical vector a sequence of integer vectors such that from them it is possible to restore a sequence of vectors with rational components, approximating the original vector. The expansions constructed here can be interpreted as a modification of continued fractions, which is called Hessenberg fractions [6], but this construction is not directly used in the algorithm.

Decomposition algorithm

Let $a_1, \ldots, a_n$ be the original vector. $Ai_1, \ldots, Ai_n$ integer parts of the vector components (they must be stored for restoration). $Ar_1, \ldots, Ar_n$ fractional parts of vector components.

We choose among the fractional parts the minimum number mi and form a new vector $b_1, \ldots, b_n$ according to the following rules:
– if $Ar_k = 0$, then $b_k = 0$,
– if $Ar_k = mi$, then $b_k = \frac{1}{mi}$,
– if $Ar_k > mi$, then $b_k = \frac{Ar_k}{mi}$.

It is easy to see that the fractional parts are reconstructed from the local bundle graph

$$Ar_k = 0 \to Ar_k = \frac{0}{0},\ Ar_k = m1 \to Ar_k = \frac{1}{b_k},\ Ar_k > mi \to Ar_k = \frac{b_k}{b_{nmi}}$$

here the number nmi is such that $b_{nmi} = \frac{1}{mi}$. The local bundle graph must be stored.

The algorithm loops: we set $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$ and repeat the procedure.

Assume the procedure is stopped at the s-th step, then from the local graphs of the bundle one can recover the global decomposition graph, moving from the top layer down (on the first layer, the local and global graphs coincide), the nodes of the graph at each level contain the numbers of integer parts from the list $Ai_1, \ldots, Ai_n$ formed for this level. The numbers must be replaced with whole parts.

If in the last level of the graph
– replace integers with the reals they were derived from,
– divide these numbers one by another, as the structure of the graph suggests,
– add the result of the division to the integers at the top level of the graph, as the graph structure suggests,
then at the penultimate level, the integer parts will be replaced by the real numbers from which the integers were obtained. If we continue the lifting procedure through the levels of the graph, then at the first level we get the original vector.

If you do not correct the last level, you get a suitable fraction. Numerical experiments show a good approximation. But the hope that the obtained suitable fractions give the closest points to the line did not materialize. The algorithm directly iterates over the nearest points to the line, gives a different set of points.

References

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Alexander Kotochigov
Department of Algorithmic Mathematics
Electrotechnical University, Russia
Saint-Petersburg, Russia

e-mail: `amkotochigov@gmail.com`