Investigation of the Stationary Motions of the System of Two Connected Bodies Moving along a Circular Orbit Using Polynomial Algebra Methods

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Abstract. Polynomial algebra methods are used to determine the equilibrium orientations of a system of two bodies connected by a spherical hinge that moves on a circular orbit. Primary attention is given to the study of equilibrium orientations of the two-body system in the plane perpendicular to the circular orbital plane. A method is proposed for transforming the system of trigonometric equations determining the equilibria into a system of polynomial equations, which in turn are reduced by calculating the resultant to a single algebraic equation of degree 12 in one unknown. By applying symbolic factorization, this algebraic equation is decomposed into three polynomial factors, each specifying a certain class of equilibrium configurations. The domains with an identical number of equilibrium positions are classified using algebraic methods for constructing a discriminant hypersurface. Using the proposed approach, it is shown that the system can have up to 48 equilibrium orientations in the plane perpendicular to the circular orbit.

Introduction

In our work, we apply polynomial algebra methods to investigate the equilibrium orientations of a system of two bodies (satellite and stabilizer) connected by a spherical hinge that moves in a central Newtonian force field along a circular orbit. Determining the equilibria for the system of bodies on a circular orbit is of practical interest for designing composite gravitational orientation systems of satellites that can stay on the orbit for a long time without energy consumption. The dynamics of various composite schemes for satellite–stabilizer gravitational orientation systems was discussed in detail in [1]. In [2], [3], [4] equilibrium orientations for the two-body system in the orbital plane were found in the case where the spherical hinge was positioned at the intersection of the principal central axes of inertia of the satellite and stabilizer, as well as in the case where the hinge was positioned on the line of intersection between two planes formed by the principal central axes of inertia of the satellite and stabilizer. In this work, we study the equilibrium orientations of the two-body system in the plane perpendicular to the circular orbital plane in the case when the hinge is positioned on the line of intersection between two planes formed by the principal central axes of inertia of the satellite and stabilizer..

1. Investigation of Equilibrium Orientations

We consider a system of two bodies connected by a spherical hinge that moves along a circular orbit [4]. To write the corresponding equations of motion, we introduce the following right-handed rectangular coordinate systems. The orbital coordinate system is $OXYZ$. The $OZ-$ axis is directed along the radius vector that connects the Earth's center of mass with the center of mass of the two-body system O, the $OX-$ axis is directed along the linear velocity vector of the center of mass O, while the $OY-$ axis is directed along the normal to the orbital plane. The coordinate system of the ith body (i= 1, 2) is $O_ix_iy_iz_i$, where the axis of these coordinate systems are the principal central axes of inertia of the ith body. The orientation of coordinate system $O_ix_iy_iz_i$ with respect to the orbital coordinate system is determined using aircraft angles [1].

Suppose that (a_i, b_i, c_i) are the coordinates of spherical hinge in the coordinate system $O_ix_iy_iz_i$; A_i, B_i, C_i are the principal central moments of inertia of the each bodies; $M = M_1M_2/(M_1 + M_2)$; M_i is the mass of the ith body.

Using the expressions of the kinetic energy of the two-body system and the force function that determines the action of the Earth's gravitational field on the two-body system in the case where $c_1 = c_2 = 0$ and its equilibrium orientations are in a plane perpendicular to the orbital plane (then, the coordinates of spherical hinge in the coordinate system of each body are given by $(a_i, b_i, 0)$) the equations of motion for this system we can written in the form of Lagrange equations of the second kind [1]. Then from Lagrange equations we can obtain the stationary trigonometric system which allows us to determine equilibrium orientations for the system of two bodies connected by the spherical hinge in the orbital coordinate system:

$$
\begin{aligned}
&\big((B_1 - A_1)/M\big) \sin x_1 \cos x_1 + (a_1 \sin x_1 + b_1 \cos x_1)(a_1 \cos x_1 - b_1 \sin x_1) - \\
&\qquad\qquad - (a_1 \cos x_1 - b_1 \sin x_1)(a_2 \sin x_2 + b_2 \cos x_2) = 0, \qquad (1)\\
&\big((B_2 - A_2)/M\big) \sin x_2 \cos x_2 + (a_2 \sin x_2 + b_2 \cos x_2)(a_2 \cos x_2 - b_2 \sin x_2) - \\
&\qquad\qquad - (a_2 \cos x_2 - b_2 \sin x_2)(a_1 \sin x_1 + b_1 \cos x_1) = 0,
\end{aligned}
$$

where x_1 and x_2 are two of the aircraft angles.

The trigonometric system (1) cannot be solved analytically for two unknown aircraft angles. To solve system (1), we use the universal approach whereby the sines and cosines of angles x_i are replaced by their tangents $t_i = \tan(x_i)$.

As a result, we obtain from (1) the algebraic system of two equations in two unknowns t_1, t_2

$$\begin{aligned}\bar{a}_0 t_1^3 + \bar{a}_1 t_1^2 + \bar{a}_2 t_1 + \bar{a}_3 = 0,\\ \bar{b}_0 t_1^2 + \bar{b}_1 t_1 + \bar{b}_2 = 0,\end{aligned} \qquad (2)$$

where $\bar{a}_i, \bar{b}_i$ are polynomials depending on six system parameters.

By using the resultant approach to eliminate t_1 from system (2) and symbolic computations in Wolfram Mathematica 12.1 to find the determinant of the resultant matrix, we obtain a twelfth-order algebraic equation in one unknown t_2, which upon factorization, turns into a product of three polynomials: $P(t_2) = P_1(t_2)P_2(t_2)P_3(t_2) = 0$. Here $P_1(t_2)$, $P_2(t_2)$ are second-order polynomials and $P_3(t_2)$ is an eighth-order polynomial, the coefficients of which are polynomials in six system parameters.

By the definition of the resultant, each root of equation $P(t_2) = 0$ corresponds to one common root of system (2) . The algebraic equation obtained has the even number of real roots, which does not exceed 12. By substituting real root of algebraic equation $P(t_2) = 0$ into the equations of system (2) , we find the common root of these equations. It can be shown that four equilibrium solutions of the original system correspond to each real root of equations (2).

Since the total number of real roots of $P(t_2) = 0$ does not exceed 12, the satellite–stabilizer system in the plane perpendicular to the orbital plane can have no more than 48 equilibrium orientations in the orbital coordinate system. Using obtained equations for each set of system parameters, we can determine all equilibrium orientations of the satellite–stabilizer system in the orbital coordinate system.

To investigate the number of equilibrium solutions for the satellite–stabilizer system, we define domains with equal numbers of real roots of $P_3(t_2) = 0$ in the space of the six parameters. For this purpose, we construct a discriminant hypersurface of this polynomial, which defines the boundary of the domains with equal numbers of real roots.

Conclusion

The use of polynomial computer algebra methods allowed us to solve the classical problem of space flight mechanics in a fairly simple form.

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