

Investigation of the Stationary Motions of the System of Two Connected Bodies Moving along a Circular Orbit Using Polynomial Algebra Methods

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1. Equations of motion

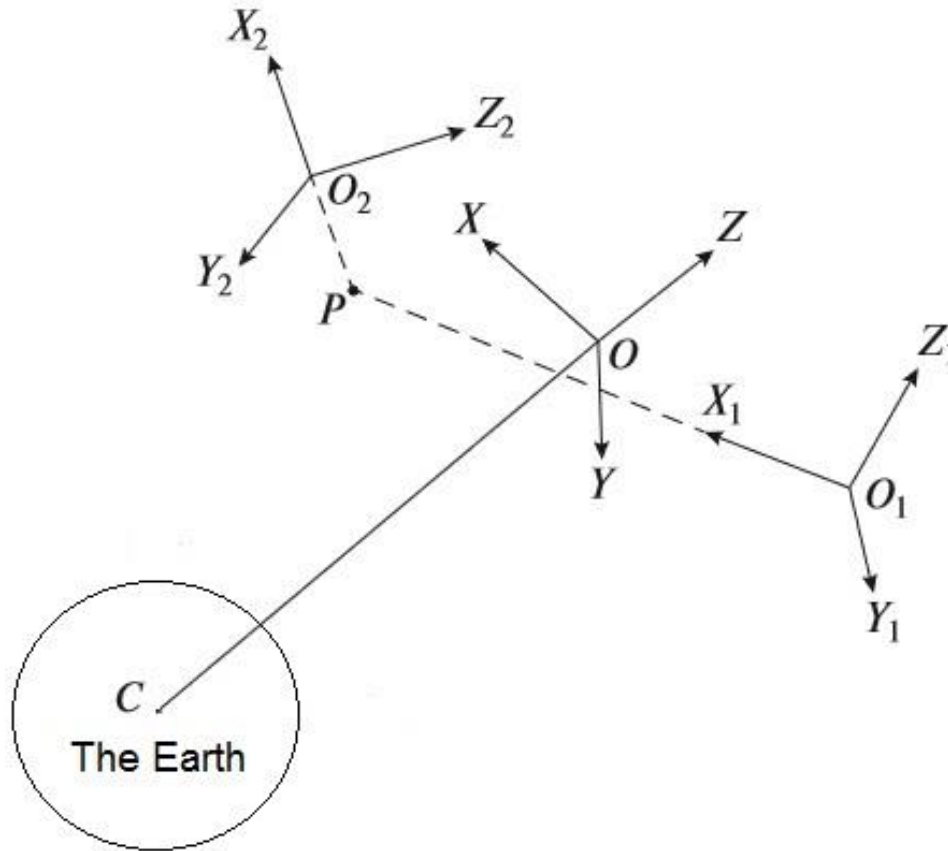


Fig. 1. Basic coordinate systems

$OXYZ$ - is the orbital coordinate system

(a_i, b_i, c_i) - are the coordinates of the

spherical hinge P in the body coordinate system $O_i x_i y_i z_i$

1. Equations of motion

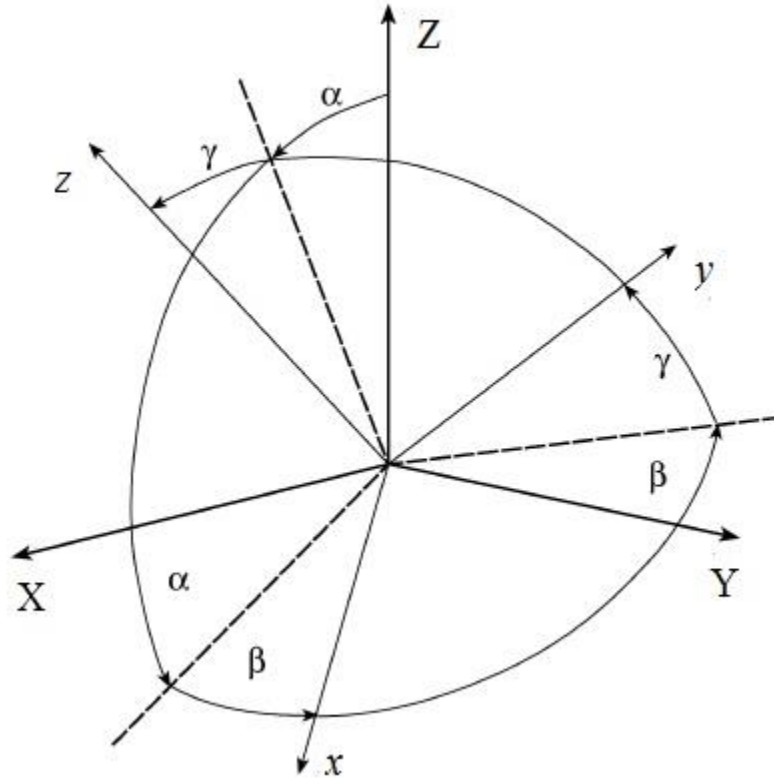


Fig. 1. Orientation of body-fixed axes with respect to the orbital coordinate system

International Space Station



International Space Station(1998-2024) in gravity orientation mode
Orbit parameters: $h_a = 418$ km, $h_p = 413$ km, $i = 51.63^\circ$

1.1 Equations of motion in the orbital plane

Consider the motion of the two bodies system around its center of mass in the plane of a circular orbit when $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, two aircraft angles $\beta_1 = \beta_2 = 0$, $\gamma_1 = \gamma_2 = 0$. The expressions for the force function, which determines the effect of the Earth gravitational field on the system of two bodies connected by a spherical hinge in the case $b_1 = b_2 = 0$ have the form:

$$\begin{aligned} U = & \frac{3}{2} M \omega_0^2 [(a_1 \sin \alpha_1 - c_1 \cos \alpha_1) - (a_2 \sin \alpha_2 - c_2 \cos \alpha_2)]^2 - \\ & - \frac{3}{2} \omega_0^2 (A_1 - C_1) \sin^2 \alpha_1 - \frac{3}{2} \omega_0^2 (A_2 - C_2) \sin^2 \alpha_2 + \\ & + M \omega_0^2 [(a_1 a_2 + c_1 c_2) \cos(\alpha_1 - \alpha_2) - (a_1 c_2 - a_2 c_1) \sin(\alpha_1 - \alpha_2)]. \end{aligned} \quad (1)$$

Here A_i, B_i, C_i are the principal central moments, $M = M_1 M_2 / (M_1 + M_2)$ and M_i is the mass of the i -th body; $\alpha_i, \beta_i, \gamma_i$ - are the angles of pitch, yaw and roll; a_{ij}, b_{ij} - the direction cosines of the axis in the orbital reference frame, $(a_i, 0, c_i)$ - are the coordinates of the of the spherical hinge of the i -th body in reference frame; ω_0 - is the angular velocity of the orbital motion of the center of mass of the two-body system.

1.1 Equations of motion

The expressions for the kinetic energy the system of two bodies connected by a hinge in the case when $b_1=b_2=0$ have the form

$$\begin{aligned} T = & \frac{1}{2}[B_1 + M(a_1^2 + c_1^2)](\dot{\alpha}_1 + \omega_0)^2 + \frac{1}{2}[B_2 + M(a_2^2 + c_2^2)](\dot{\alpha}_2 + \omega_0)^2 - \\ & - M[(a_1 a_2 + c_1 c_2) \cos(\alpha_1 - \alpha_2) - \\ & - (a_1 c_2 - a_2 c_1) \sin(\alpha_1 - \alpha_2)](\dot{\alpha}_1 + \omega_0)(\dot{\alpha}_2 + \omega_0). \end{aligned} \quad (2)$$

The equations of motion for this system can be written in Lagrange form the second kind by symbolic differentiation D in the Wolfram *Mathematica* 12.1

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{\alpha}_i} - \frac{\partial T}{\partial \alpha_i} - \frac{\partial U}{\partial \alpha_i} = 0, \quad i = 1, 2.$$

1.1 Equations of motion

The Lagrange equations have the form

$$\begin{aligned} & (B_1 + M(a_1^2 + c_1^2))\ddot{\alpha}_1 - M((a_1a_2 + c_1c_2)\cos(\alpha_1 - \alpha_2) - \\ & -(a_1c_2 - a_2c_1)\sin(\alpha_1 - \alpha_2))\ddot{\alpha}_2 - M((a_1a_2 + c_1c_2)\sin(\alpha_1 - \alpha_2) + \\ & +(a_1c_2 - a_2c_1)\cos(\alpha_1 - \alpha_2))(\dot{\alpha}_2^2 + 2\dot{\alpha}_2) + 3(A_1 - C_1)\sin\alpha_1\cos\alpha_1 - \\ & -3M(a_1\cos\alpha_1 + c_1\sin\alpha_1)((a_1\sin\alpha_1 - c_1\cos\alpha_1) - (a_2\sin\alpha_2 - c_2\cos\alpha_2)) = 0, \end{aligned} \quad (3)$$

$$\begin{aligned} & -M((a_1a_2 + c_1c_2)\cos(\alpha_1 - \alpha_2) - (a_1c_2 - a_2c_1)\sin(\alpha_1 - \alpha_2))\ddot{\alpha}_1 + \\ & +(B_2 + M(a_2^2 + c_2^2))\ddot{\alpha}_2 + 3(A_2 - C_2)\sin\alpha_2\cos\alpha_2 + \\ & +M((a_1a_2 + c_1c_2)\sin(\alpha_1 - \alpha_2) + (a_1c_2 - a_2c_1)\cos(\alpha_1 - \alpha_2))(\dot{\alpha}_1^2 + 2\dot{\alpha}_1) + \\ & +3M(a_2\cos\alpha_2 + c_2\sin\alpha_2)((a_1\sin\alpha_1 - c_1\cos\alpha_1) - (a_2\sin\alpha_2 - c_2\cos\alpha_2)) = 0. \end{aligned} \quad (4)$$

1.2 Equilibrium orientations

By assuming in (3) - (4) $\alpha_i = \text{const}$ we obtain the stationary equations

$$\begin{aligned} & ((A_1 - C_1)/M) \sin \alpha_1 \cos \alpha_1 - (a_1 \cos \alpha_1 + c_1 \sin \alpha_1)(a_1 \sin \alpha_1 - c_1 \cos \alpha_1) + \\ & + (a_1 \cos \alpha_1 + c_1 \sin \alpha_1)(a_2 \sin \alpha_2 - c_2 \cos \alpha_2) = 0, \end{aligned} \tag{5}$$

$$\begin{aligned} & ((A_2 - C_2)/M) \sin \alpha_2 \cos \alpha_2 - (a_2 \cos \alpha_2 + c_2 \sin \alpha_2)(a_2 \sin \alpha_2 - c_2 \cos \alpha_2) + \\ & + (a_2 \cos \alpha_2 + c_2 \sin \alpha_2)(a_1 \sin \alpha_1 - c_1 \cos \alpha_1) = 0. \end{aligned}$$

Which allow us to determine the equilibrium orientation for the system of two bodies connected by a spherical hinge in the orbital coordinate system the plane of a circular orbit .

Equations (5) form a closed system of two equations with respect to the two aircraft angles.

Trigonometric system (5) cannot be solved directly.

1.3 Investigations of Equilibria

For system (5) we used the universal change of sines and cosines through the tangent $t_i = \tan(\alpha_i)$

$$\sin \alpha_i = \frac{\tan(\alpha_i)}{\sqrt{1 + \tan^2(\alpha_i)}} = \frac{t_i}{\sqrt{1 + t_i^2}}, \quad \cos \alpha_i = \frac{1}{\sqrt{1 + \tan^2(\alpha_i)}} = \frac{1}{\sqrt{1 + t_i^2}}.$$

and obtain from (5) two equations with two unknowns t_1 and t_2

$$d_1 t_1 - (a_1 + c_1 t_1)(a_1 t_1 - c_1) = (a_1 + c_1 t_1)(c_2 - a_2 t_2) \frac{\sqrt{1 + t_1^2}}{\sqrt{1 + t_2^2}}, \quad (6)$$

$$d_2 t_2 - (a_2 + c_2 t_2)(a_2 t_2 - c_2) = (a_2 + c_2 t_2)(c_1 - a_1 t_1) \frac{\sqrt{1 + t_2^2}}{\sqrt{1 + t_1^2}}.$$

Where $d_1 = (A_1 - C_1)/M$, $d_2 = (A_2 - C_2)/M$.

Then, we first divide and second multiply the left-hands and the right-hands sides of these equations and obtain two algebraic equations with two unknowns t_1 and t_2 .

1.3 Investigations of Equilibria

$$\begin{aligned}\bar{a}_0 t_1^3 + \bar{a}_1 t_1^2 + \bar{a}_2 t_1 + a_3 &= 0, \\ \bar{b}_0 t_1^2 + \bar{b}_1 t_1 + \bar{b}_2 &= 0.\end{aligned}\tag{7}$$

Where

$$\begin{aligned}\bar{a}_0 &= c_1(c_2(a_1^2 - a_2^2)t_2^3 + a_2(a_1^2 - a_2^2 + 2c_2^2 + d_2)t_2^2 + \\ &+ c_2(a_1^2 - c_2^2 + 2a_2^2 - d_2)t_2 + a_2(a_1^2 - c_2^2)),\end{aligned}$$

$$\begin{aligned}\bar{a}_1 &= a_1(c_2(a_1^2 - a_2^2 - 2c_1^2 - d_1)t_2^3 + a_2(a_1^2 - a_2^2 - 2c_1^2 + 2c_2^2 - d_1 + d_2)t_2^2 + \\ &+ c_2(a_1^2 + 2a_2^2 - 2c_1^2 - c_2^2 - d_1 - d_2)t_2 + a_2(a_1^2 - 2c_1^2 - c_2^2 - d_1)),\end{aligned}$$

$$\begin{aligned}\bar{a}_2 &= c_1(c_2(c_1^2 - a_2^2 - 2a_1^2 + d_1)t_2^3 + a_2(c_1^2 - a_2^2 - 2a_1^2 + 2c_2^2 + d_1 + d_2)t_2^2 - \\ &- c_2(2a_1^2 - 2a_2^2 - c_1^2 + c_2^2 - d_1 + d_2)t_2 - a_2(2a_1^2 - c_1^2 + c_2^2 - d_1)),\end{aligned}$$

$$\begin{aligned}\bar{a}_3 &= a_1(c_2(c_1^2 - a_2^2)t_2^3 - a_2(a_2^2 - c_1^2 - 2c_2^2 - d_2)t_2^2 + \\ &+ c_2(2a_2^2 + c_1^2 - c_2^2 - d_2)t_2 + a_2(c_1^2 - c_2^2)),\end{aligned}$$

1.3 Investigations of Equilibria

$$\bar{b}_0 = a_1 c_1 d_2 t_2,$$

$$\bar{b}_1 = a_2 c_2 d_1 t_2^2 + ((a_2^2 - c_2^2)d_1 + (a_1^2 - c_1^2)d_2 - d_1 d_2)t_2 - a_2 c_2 d_1, \quad (8)$$

$$\bar{b}_2 = -a_1 c_1 d_2 t_2.$$

Here

$$d_1 = \frac{(A_1 - C_1)}{M}, \quad d_2 = \frac{(A_2 - C_2)}{M}.$$

1.3 Investigations of Equilibria

Using the Resultant concept we eliminate the variable t_1 from Eq.(7). Expanding the determinant of resultant matrix of Eq.(7) with the help of *Mathematica* matrix function **Resultant**, we obtain the 12th order algebraic equation in t_2 . After factorization this equation has the form

$$P_1(t_2)P_2(t_2)P_3(t_2) = 0. \quad (9)$$

Here

$$P_1(t_2) = a_1 c_1 d_1 (a_2 t_2 - c_2)^2 = 0,$$

$$P_2(t_2) = a_2 c_2 t_2^2 + (a_2^2 - c_2^2 + d_2) t_2 + a_2 c_2 = 0,$$

$$P_3(t_2) = p_0 t_2^8 + p_1 t_2^7 + p_2 t_2^6 + p_3 t_2^5 + p_4 t_2^4 + p_5 t_2^3 + p_6 t_2^2 + p_7 t_2 + p_8,$$

$$p_0 = c_2^4 d_1^2 (a_1^2 - a_2^2)(a_2^2 - c_1^2), \quad (10)$$

$$p_1 = 2a_2 c_2^3 d_1 (d_2 (a_1^2 - c_1^2)(a_1^2 - a_2^2 + c_1^2) + d_1 (2a_2^2 (a_1^2 - a_2^2) + c_1^2 (c_2^2 - a_1^2 - d_2) + (2a_2^2 - a_1^2)(c_1^2 + c_2^2 + d_2))), \dots$$

Here p_i — are rather complicated coefficients, depending on 6 parameters. Using Eqs.(9) and (7), for each set of the system parameters, we can determine numerically the angles α_2 and α_1 , that is, all the equilibrium orientations of the satellite–stabilizer system.

1.3 Investigations of Equilibria

In studying the two-body system equilibrium orientations, we determine the domains with an equal number of real roots of Eq.(9) in the space of 6 parameters. The decomposition of the space of parameters into domains with an equal number of real roots is determined by the discriminant hypersurface. The form of the discriminant of the polynomial $P_3(t_2)$ is a very cumbersome expression.

Let us consider a simpler case when $a_1 = a_2 = c_1 = c_2 = a$.

In this case from system (10) we will obtain the equation of the 4th degree

$$P_{10}(t_2) = p_{00}t_2^4 + p_{01}t_2^3 + p_{02}t_2^2 + p_{03}t_2 + p_{04} = 0, \quad (11)$$

where

$$p_{00} = 4(d_{01}d_{02} + d_{01} - d_{02})(d_{01}d_{02} + d_{01} + d_{02}),$$

$$p_{01} = -4(d_{02} + 2)(d_{01}^2d_{02}^2 - d_{01}^2d_{02} - 2d_{01}^2 + 2d_{02}^2),$$

$$p_{02} = (d_{01}^2d_{02}^4 + 4d_{02}^4 - 16d_{01}^2d_{02}^2 + 24(d_{01}^2 - d_{02}^2)),$$

$$p_{03} = 4(d_{02} - 2)(d_{01}^2d_{02}^2 + d_{01}^2d_{02} - 2d_{01}^2 + 2d_{02}^2),$$

$$p_{04} = 4(d_{01}d_{02} - d_{01} - d_{02})(d_{01}d_{02} - d_{01} + d_{02}).$$

1.3 Investigations of Equilibria

Here

$$d_{01} = (A_1 - C_1) / M a^2,$$

$$d_{02} = (A_2 - C_2) / M a^2;$$

Discriminant hypersurface of the polynomial (11) $P_1(t_2)$ (resultant of the two polynomials $P_1(t_2)$ and $P'_1(t_2)$) has the form

$$\begin{aligned} P_5(d_{01}, d_{02}) &= \\ &= 1024 d_{02}^{12} P_6(d_{01}, d_{02}) P_7(d_{01}, d_{02}) P_8(d_{01}, d_{02}) = 0. \end{aligned}$$

Here

(12)

$$P_6(d_{01}, d_{02}) = (d_{01}^2 (d_{02} + 1)^2 - d_{02}^2),$$

$$P_7(d_{01}, d_{02}) = ((d_{01}^2 d_{02}^2 + 4(d_{01} - d_{02})^2)((d_{01}^2 d_{02}^2 + 4(d_{01} + d_{02})^2)),$$

$$P_8(d_{01}, d_{02}) = (d_{01}^4 (d_{02}^2 + 8)^2 + 16 d_{01}^2 d_{02}^2 (d_{02}^2 + 10) + 64 (d_{02}^4 - 108)).$$

1.3 Investigations of Equilibria

$P_6(d_{01}, d_{02})$ and $P_7(d_{01}, d_{02})$ are the polynomials of the 4th and 8th degree.

Now we should check the change in the number of equilibria when the curve (12) is intersected. This can be done numerically by determining the number of equilibria at a single point of each domain at the plane (d_{01}, d_{02}) .

Only the curve $P_8(d_{01}, d_{02}) = 0$ separates the domains with different number of equilibria.

Fig.2 shows the distributions of domains with equal number of real roots of Eq.(11) and indicates the domains where 4 and 2 real solutions exist (16 and 8 equilibrium orientations).

In case when $a_1 = a_2 = c_1 = c_2 = a$

$$P_1(t_2) = a_1 c_1 d_1 (a_2 t_2 - c_2)^2 = a^3 d_1 (t_2 - 1)^2 = 0; \quad t_2 = \tan \alpha_2 = 1$$

$$P_2(t_2) = a_2 c_2 t_2^2 + (a_2^2 - c_2^2 + d_2) t_2 + a_2 c_2 = t_2^2 + d_{02} t_2 + 1 = 0,$$

$$t_2 = (-d_{02} \pm \sqrt{d_{02}^2 - 4}) / 2.$$

Therefore, in the case when $a_1 = a_2 = c_1 = c_2 = a$, and $|d_{02}| \leq 2$ there exist only 28 and 20 equilibrium orientations for the satellite–stabilizer system in the plane of a circular orbit.

1.3 Investigations of Equilibria

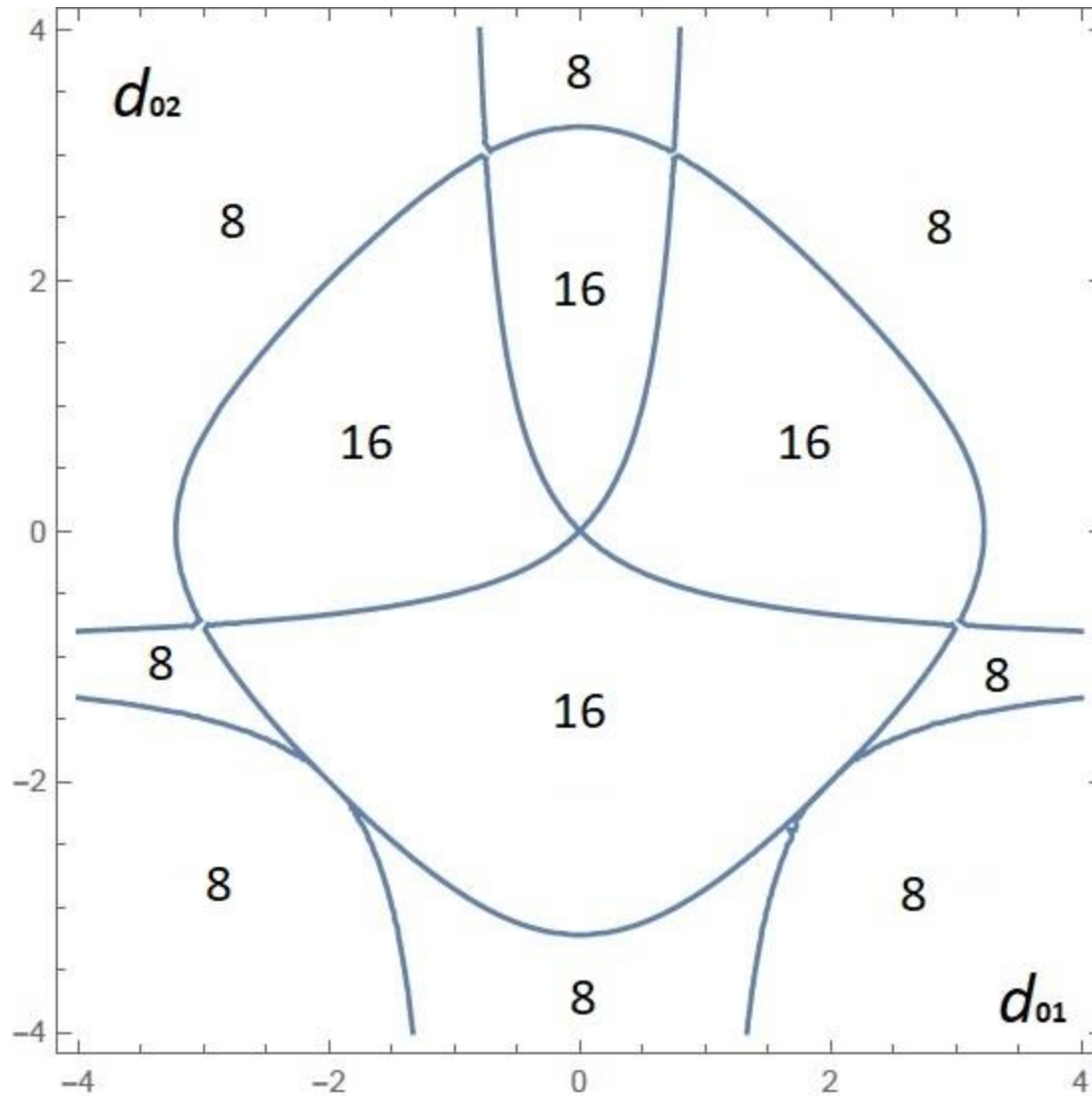


Fig. 2. The regions with the fixed number of equilibria

2.1. Equations of motion in the plane tangent to orbital plane

Consider the motion of the two bodies system around its center of mass in the plane perpendicular to the orbital plane of a circular orbit when

$\beta_1 \neq 0, \beta_2 \neq 0$ and two aircraft angles $\alpha_1 = \alpha_2 = 0, \gamma_1 = \gamma_2 = 0$.

The expressions for the force function, which determines the effect of the Earth gravitational field on the system of two bodies connected by a spherical hinge in the case $c_1 = c_2 = 0$ have the form:

$$U = M \omega_0^2 ((a_1 a_2 + b_1 b_2) \cos(\beta_1 - \beta_2) + (a_1 b_2 - a_2 b_1) \sin(\beta_1 - \beta_2)). \quad (2.1)$$

Here A_i, B_i, C_i are the principal central moments, $M = M_1 M_2 / (M_1 + M_2)$ and M_i is the mass of the i -th body; $\alpha_i, \beta_i, \gamma_i$ - are the angles of pitch, yaw and roll; a_{ij}, b_{ij} - the direction cosines of the axis in the orbital reference frame, $(a_i, b_i, 0)$ - are the coordinates of the of the spherical hinge of the i -th body in reference frame; ω_0 - is the angular velocity of the orbital motion of the center of mass of the two-body system.

2.1. Equations of motion

The expressions for the kinetic energy the system of two bodies connected by a hinge in the case when $c_1=c_2=0$ have the form

$$\begin{aligned} T = & \frac{1}{2}(C_1 + M(a_1^2 + b_1^2))\dot{\beta}_1^2 + \frac{1}{2}(C_2 + M(a_2^2 + b_2^2))\dot{\beta}_2^2 - \\ & M((a_1a_2 + b_1b_2)\cos(\beta_1 - \beta_2) + (a_1b_2 - a_2b_1)\sin(\beta_1 - \beta_2))\dot{\beta}_1\dot{\beta}_2 + \\ & + \frac{1}{2}((B_1 - A_1) - M(b_1^2 - a_1^2))\omega_0^2 \cos^2 \beta_1 - Ma_1a_2\omega_0^2 \cos \beta_1 \cos \beta_2 + \\ & + \frac{1}{2}((B_2 - A_2) - M(b_2^2 - a_2^2))\omega_0^2 \cos^2 \beta_2 - Mb_1b_2\omega_0^2 \sin \beta_1 \sin \beta_2 - \\ & - M\omega_0^2(b_1 \sin \beta_1 - b_2 \sin \beta_2)(a_1 \cos \beta_1 - a_2 \cos \beta_2). \end{aligned} \quad (2.2)$$

The equations of motion for this system can be written in Lagrange form the second kind by symbolic differentiation D in the Wolfram *Mathematica* 12.1

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{\alpha}_i} - \frac{\partial T}{\partial \alpha_i} - \frac{\partial U}{\partial \alpha_i} = 0, \quad i = 1, 2.$$

2.1. Equations of motion

The Lagrange equations have the form

$$\begin{aligned} & (C_1 + M(a_1^2 + b_1^2))\ddot{\beta}_1 - M((a_1a_2 + b_1b_2)\cos(\beta_1 - \beta_2) + \\ & + (a_1b_2 - a_2b_1)\sin(\beta_1 - \beta_2))\ddot{\beta}_2 - M((a_1a_2 + b_1b_2)\sin(\beta_1 - \beta_2) \\ & - (a_1b_2 - a_2b_1)\cos(\beta_1 - \beta_2))\dot{\beta}_2^2 + (B_1 - A_1)\omega_0^2 \sin \beta_1 \cos \beta_1 + \\ & + M\omega_0^2(a_1 \sin \beta_1 + b_1 \cos \beta_1)(a_1 \cos \beta_1 - b_1 \sin \beta_1) - \\ & - M\omega_0^2(a_1 \cos \beta_1 - b_1 \sin \beta_1)(a_2 \sin \beta_2 + b_2 \cos \beta_2) = 0, \end{aligned} \quad (2.3)$$

$$\begin{aligned} & (C_2 + M(a_2^2 + b_2^2))\ddot{\beta}_2 - M((a_1a_2 + b_1b_2)\cos(\beta_1 - \beta_2) + \\ & + (a_1b_2 - a_2b_1)\sin(\beta_1 - \beta_2))\ddot{\beta}_1 + M((a_1a_2 + b_1b_2)\sin(\beta_1 - \beta_2) - \\ & - (a_1b_2 - a_2b_1)\cos(\beta_1 - \beta_2))\dot{\beta}_1^2 + (B_2 - A_2)\omega_0^2 \sin \beta_2 \cos \beta_2 + \\ & + M\omega_0^2(a_2 \sin \beta_2 + b_2 \cos \beta_2)(a_2 \cos \beta_2 - b_2 \sin \beta_2) - \\ & - M\omega_0^2(a_2 \cos \beta_2 - b_2 \sin \beta_2)(a_1 \sin \beta_1 + b_1 \cos \beta_1) = 0. \end{aligned} \quad (2.4)$$

2.2 Equilibrium orientations

By assuming in (2.3) - (2.4) $\beta_i = \text{const}$ we obtain the stationary equations

$$\begin{aligned} & ((B_1 - A_1)/M) \sin \beta_1 \cos \beta_1 + (a_1 \sin \beta_1 + b_1 \cos \beta_1)(a_1 \cos \beta_1 - b_1 \sin \beta_1) - \\ & - (a_1 \cos \beta_1 - b_1 \sin \beta_1)(a_2 \sin \beta_2 + b_2 \cos \beta_2) = 0, \end{aligned} \tag{2.5}$$

$$\begin{aligned} & ((B_2 - A_2)/M) \sin \beta_2 \cos \beta_2 + (a_2 \sin \beta_2 + b_2 \cos \beta_2)(a_2 \cos \beta_2 - b_2 \sin \beta_2) - \\ & - (a_2 \cos \beta_2 - b_2 \sin \beta_2)(a_1 \sin \beta_1 + b_1 \cos \beta_1) = 0. \end{aligned}$$

Which allow us to determine the equilibrium orientation for the system of two bodies connected by a spherical hinge in the orbital coordinate system the plane perpendicular to the circular orbit .

Equations (2.5) form a closed system of two equations with respect to the two aircraft angles.

2.3. Investigations of Equilibria

For system (2.5) we used the universal change of sines and cosines through the tangent $t_i = \tan(\beta_i)$

$$\sin \alpha_i = \frac{\tan(\beta_i)}{\sqrt{1 + \tan^2(\beta_i)}} = \frac{t_i}{\sqrt{1 + t_i^2}}, \quad \cos \beta_i = \frac{1}{\sqrt{1 + \tan^2(\beta_i)}} = \frac{1}{\sqrt{1 + t_i^2}}.$$

and obtain from (5) two equations with two unknowns t_1 and t_2

$$\begin{aligned} & ((B_1 - A_1)/M) \operatorname{tg} \beta_1 + (a_1 \operatorname{tg} \beta_1 + b_1)(a_1 - b_1 \operatorname{tg} \beta_1) - \\ & - (a_1 - b_1 \operatorname{tg} \beta_1)(a_2 \operatorname{tg} \beta_2 + b_2) \frac{\sqrt{1 + \operatorname{tg}^2 \beta_1}}{\sqrt{1 + \operatorname{tg}^2 \beta_2}} = 0, \end{aligned} \quad (2.6)$$

$$\begin{aligned} & ((B_2 - A_2)/M) \operatorname{tg} \beta_2 + (a_2 \operatorname{tg} \beta_2 + b_2)(a_2 - b_2 \operatorname{tg} \beta_2) - \\ & - (a_2 - b_2 \operatorname{tg} \beta_2)(a_1 \operatorname{tg} \beta_1 + b_1) \frac{\sqrt{1 + \operatorname{tg}^2 \beta_2}}{\sqrt{1 + \operatorname{tg}^2 \beta_1}} = 0. \end{aligned}$$

Then, we first divide and second multiply the left-hands and the right-hands sides of these equations and obtain two algebraic equations with two unknowns t_1 and t_2 .

2.3. Investigations of Equilibria

$$\begin{aligned}\bar{a}_0 t_1^3 + \bar{a}_1 t_1^2 + \bar{a}_2 t_1 + a_3 &= 0, \\ \bar{b}_0 t_1^2 + \bar{b}_1 t_1 + \bar{b}_2 &= 0.\end{aligned}\tag{2.7}$$

Where

$$\bar{a}_0 = b_1((a_2 t_2 + b_2)(a_2 b_2 + (d_2 + a_2^2 - b_2^2)t_2 - a_2 b_2 t_2^2) - a_1^2(a_2 - b_2 t_2)(1 + t_2^2)),$$

$$\begin{aligned}\bar{a}_1 = -a_1((a_2 t_2 + b_2) &\left(a_2 b_2 + (d_2 + a_2^2 - b_2^2)(t_2) - a_2 b_2 t_2^2 \right) - \\ &-(d_1 + a_1^2 - 2b_1^2)(a_2 - b_2 t_2)(1 + t_2^2)),\end{aligned}$$

$$\begin{aligned}\bar{a}_2 = b_1((a_2 t_2 + b_2)(a_2 b_2 + (d_2 + a_2^2 - b_2^2) &(t_2 - a_2 b_2 t_2^2)) + \\ &+(d_1 + 2a_1^2 - b_1^2)(a_2 - b_2 t_2)(1 + t_2^2)),\end{aligned}$$

$$\begin{aligned}\bar{a}_3 = -a_1((a_2 t_2 + b_2) &(a_2 b_2 + (d_2 + a_2^2 - b_2^2)t_2 - a_2 b_2 t_2^2) - \\ &- b_1^2(a_2 - b_2 t_2)(1 + t_2^2)),\end{aligned}$$

2.3. Investigations of Equilibria

$$\bar{b}_0 = a_1 b_1 d_2 t_2,$$

$$\bar{b}_1 = a_2 b_2 d_1 t_2^2 - d_1 d_2 t_2 + (b_1^2 - a_1^2) d_2 t_2 + (b_2^2 - a_2^2) d_1 t_2 - a_2 b_2 d_1, \quad (2.8)$$

$$\bar{b}_2 = -a_1 b_1 d_2 t_2.$$

Here

$$d_1 = \frac{(B_1 - A_1)}{M}, \quad d_2 = \frac{(B_2 - A_2)}{M}.$$

2.3. Investigations of Equilibria

Using the Resultant concept we eliminate the variable t_1 from Eq.(2.7). Expanding the determinant of resultant matrix of Eq.(2.7) with the help of *Mathematica* matrix function **Resultant**, we obtain the 12th order algebraic equation in t_2 . After factorization this equation has the form

$$P_1(t_2)P_2(t_2)P_3(t_2) = 0. \quad (2.9)$$

Here

$$P_1(t_2) = a_1 b_1 d_1 (b_2 + a_2 t_2)^2 = 0,$$

$$P_2(t_2) = a_2 b_2 + (a_2^2 - b_2^2 + d_2) t_2 - a_2 b_2 t_2^2 = 0,$$

$$P_3(t_2) = p_0 t_2^8 + p_1 t_2^7 + p_2 t_2^6 + p_3 t_2^5 + p_4 t_2^4 + p_5 t_2^3 + p_6 t_2^2 + p_7 t_2 + p_8,$$

$$p_0 = d_1^2 b_2^4 (a_1^2 - a_2^2)(a_2^2 - b_1^2), \quad (2.10)$$

$$p_1 = -2a_2 b_2^3 d_1 (d_1 (2a_1^2 a_2^2 - 2a_2^4 - 2a_1^2 b_1^2 + 2a_2^2 b_1^2 - a_1^2 b_2^2 + 2a_2^2 b_2^2 - b_1^2 b_2^2) \\ + d_2 (a_1^4 - a_1^2 a_2^2 + a_2^2 b_1^2 - b_1^4) + d_1 d_2 ((a_1^2 - 2a_2^2 + b_1^2))), \dots$$

Here p_i — are rather complicated coefficients, depending on 6 parameters. Using Eqs.(2.9) and (2.7), for each set of the system parameters, we can determine numerically the angles β_2 and β_1 , that is, all the equilibrium orientations of the satellite–stabilizer system.

2.3. Investigations of Equilibria

The form of the discriminant of the polynomial $P_3(t_2)$ is a very cumbersome expression.

Let us consider a simple case when $a_1 = a_2 = b_1 = b_2 = b$.

In this case from system (2.10) we will obtain the equation of the 4th degree

$$P_{10}(t_2) = p_{00}t_2^4 + p_{01}t_2^3 + p_{02}t_2^2 + p_{03}t_2 + p_{04} = 0,$$

where

$$\begin{aligned} p_{00} &= 4(d_{01}d_{02} + d_{01} - d_{02})(d_{01}d_{02} + d_{01} + d_{02}), \\ p_{01} &= -4(d_{02} + 2)(d_{01}^2d_{02}^2 - d_{01}^2d_{02} - 2d_{01}^2 + 2d_{02}^2), \\ p_{02} &= (d_{01}^2d_{02}^4 + 4d_{02}^4 - 16d_{01}^2d_{02}^2 + 24(d_{01}^2 - d_{02}^2)), \\ p_{03} &= 4(d_{02} - 2)(d_{01}^2d_{02}^2 + d_{01}^2d_{02} - 2d_{01}^2 + 2d_{02}^2), \\ p_{04} &= 4(d_{01}d_{02} - d_{01} - d_{02})(d_{01}d_{02} - d_{01} + d_{02}). \end{aligned} \tag{2.11}$$

2.3. Investigations of Equilibria

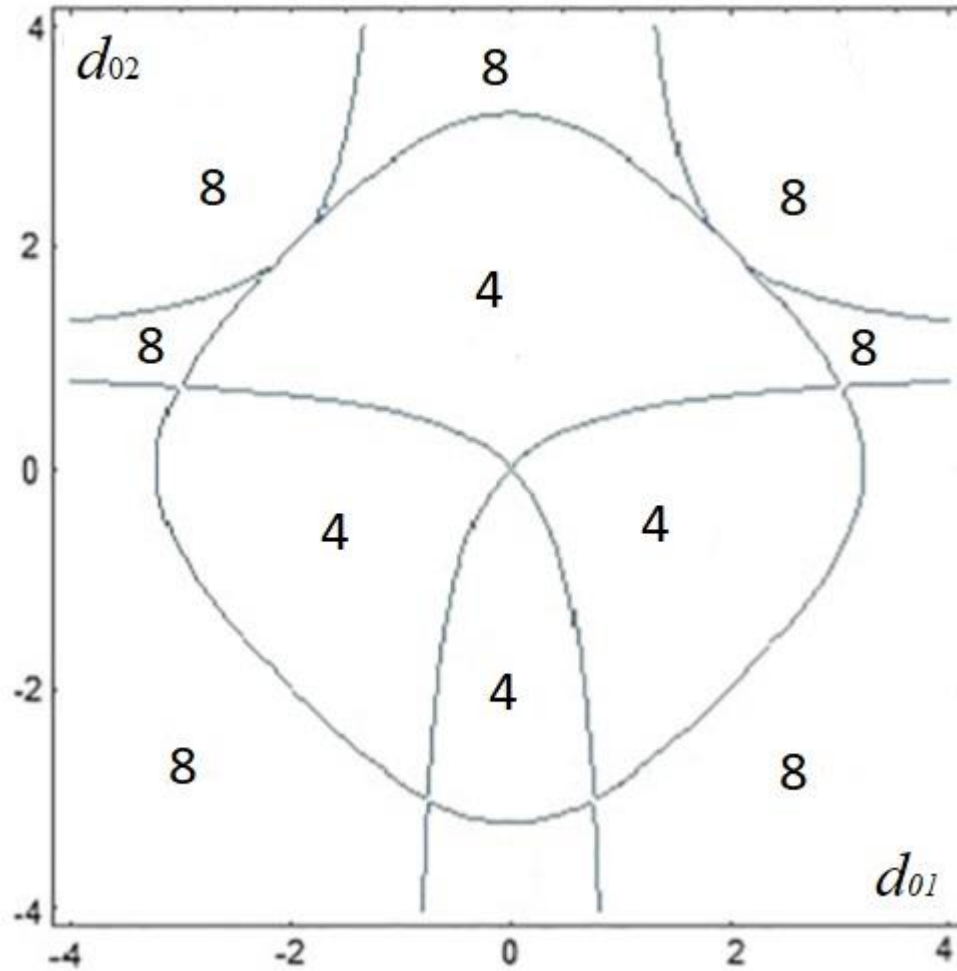


Fig. 2.2 The regions with the fixed number of equilibria

3.1. Equations of motion in the plane perpendicular to a circular orbit

Consider the motion of the two bodies system around its center of mass in the plane perpendicular to a circular orbit when two aircraft angles $\alpha_1 = \alpha_2 = 0$ $\beta_1 = \beta_2 = 0$ and $\gamma_1 \neq 0, \gamma_2 \neq 0$. The expressions for the force function, which determines the effect of the Earth's gravitational field on the system of two bodies connected by a spherical hinge in the case $a_1 = a_2 = 0$ have the form:

$$\begin{aligned}
 U = & M \omega_0^2 [(b_1 b_2 + c_1 c_2) \cos(\gamma_1 - \gamma_2) + (b_1 c_2 - b_2 c_1) \sin(\gamma_1 - \gamma_2)] + \\
 & \frac{3}{2} M \omega_0^2 [(b_1 \sin \gamma_1 - b_2 \sin \gamma_2) + (c_1 \cos \gamma_1 - c_2 \cos \gamma_2)]^2 + \\
 & + \frac{3}{2} \omega_0^2 [(B_1 - C_1) \cos^2 \gamma_1 + \frac{3}{2} \omega_0^2 [(B_2 - C_2) \cos^2 \gamma_2].
 \end{aligned} \tag{3.1}$$

Here A_i, B_i, C_i are the principal central moments, $M = M_1 M_2 / (M_1 + M_2)$ and M_i is the mass of the i -th body, p_i, q_i, r_i are the projections of the angular velocity of the i -th body on the axes $O_i x_i, O_i y_i, O_i z_i$; $\alpha_i, \beta_i, \gamma_i$ - are the angles of pitch, yaw and roll; a_{ij}, b_{ij} - the direction cosines of the axis in the orbital reference frame, $(0, b_i, c_i)$ - are the coordinates of the of the spherical hinge of the i -th body in reference frame; ω_0 - is the angular velocity of the orbital motion of the for the center of mass of the two-body system.

3.1. Equations of motion

The expressions for the kinetic energy the system of two bodies connected by a hinge in the case when $a_1=a_2=0$ have the form

$$\begin{aligned} T = & \frac{1}{2}[A_1 + M(b_1^2 + c_1^2)]\dot{\gamma}_1^2 + \frac{1}{2}[A_2 + M(b_2^2 + c_2^2)]\dot{\gamma}_2^2 + \\ & + \frac{1}{2}[(B_1 - C_1) - M(b_1^2 - c_1^2)]\omega_0^2 \cos^2 \gamma_1 + \frac{1}{2}[(B_2 - C_2) - M(b_2^2 - c_2^2)]\omega_0^2 \cos^2 \gamma_2 - \\ & - M[(b_1 b_2 + c_1 c_2) \cos(\gamma_1 - \gamma_2) + (b_1 c_2 - b_2 c_1) \sin(\gamma_1 - \gamma_2)]\dot{\gamma}_1 \dot{\gamma}_2 - \\ & - M b_1 b_2 \omega_0^2 \sin \gamma_1 \sin \gamma_2 - M c_1 c_2 \omega_0^2 \cos \gamma_1 \cos \gamma_2 + \\ & + M \omega_0^2 (b_1 \sin \gamma_1 - b_2 \sin \gamma_2)(c_1 \cos \gamma_1 - c_2 \cos \gamma_2). \end{aligned} \tag{3.2}$$

The equations of motion for this system can be written in Lagrange form the second kind by symbolic differentiation in the *Mathematica*

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{\gamma}_i} - \frac{\partial T}{\partial \gamma_i} - \frac{\partial U}{\partial \gamma_i} = 0, \quad i = 1, 2.$$

3.1. Equations of motion

The Lagrange equations have the form

$$\begin{aligned} & [A_1 + M(b_1^2 + c_1^2)]\ddot{\gamma}_1 - \\ & -M[(b_1b_2 + c_1c_2)\cos(\gamma_1 - \gamma_2) + (b_1c_2 - b_2c_1)\sin(\gamma_1 - \gamma_2)]\ddot{\gamma}_2 - \\ & -M[(b_1b_2 + c_1c_2)\sin(\gamma_1 - \gamma_2) - (b_1c_2 - b_2c_1)\cos(\gamma_1 - \gamma_2)]\dot{\gamma}_2^2 + \\ & +4[(B_1 - C_1) - M(b_1^2 - c_1^2)]\omega_0^2 \sin \gamma_1 \cos \gamma_1 - 4Mb_1c_1\omega_0^2 \cos 2\gamma_1 + \\ & +M\omega_0^2(b_1 \sin \gamma_1 + c_1 \cos \gamma_1)(b_2 \cos \gamma_2 - c_2 \sin \gamma_2) + \\ & +3M\omega_0^2(b_1 \cos \gamma_1 - c_1 \sin \gamma_1)(b_2 \sin \gamma_2 + c_2 \cos \gamma_2) = 0, \end{aligned} \quad (3.3)$$

$$\begin{aligned} & [A_2 + M(b_2^2 + c_2^2)]\ddot{\gamma}_2 - \\ & -M[(b_1b_2 + c_1c_2)\cos(\gamma_1 - \gamma_2) + (b_1c_2 - b_2c_1)\sin(\gamma_1 - \gamma_2)]\dot{\gamma}_1 + \\ & +M[(b_1b_2 + c_1c_2)\sin(\gamma_1 - \gamma_2) - (b_1c_2 - b_2c_1)\cos(\gamma_1 - \gamma_2)]\dot{\gamma}_1^2 + \\ & +4[(B_2 - C_2) - M(b_2^2 - c_2^2)]\omega_0^2 \sin \gamma_2 \cos \gamma_2 - 4Mb_2c_2\omega_0^2 \cos 2\gamma_2 + \\ & +M\omega_0^2(b_2 \sin \gamma_2 + c_2 \cos \gamma_2)(b_1 \cos \gamma_1 - c_1 \sin \gamma_1) + \\ & +3M\omega_0^2(b_2 \cos \gamma_2 - c_2 \sin \gamma_2)(b_1 \sin \gamma_1 + c_1 \cos \gamma_1) = 0. \end{aligned} \quad (3.4)$$

3.2 Equilibrium orientations

By assuming in (3.3) - (3.4) $\gamma_i = \text{const}$ we obtain the stationary equations

$$\begin{aligned} &4[(C_1 - B_1)/M + b_1^2 - c_1^2]\sin \gamma_1 \cos \gamma_1 + b_1 c_1 (\cos^2 \gamma_1 - \sin^2 \gamma_1) - \\ &-(b_1 \sin \gamma_1 + c_1 \cos \gamma_1)(b_2 \cos \gamma_2 - c_2 \sin \gamma_2) - \\ &-3(b_1 \cos \gamma_1 - c_1 \sin \gamma_1)(b_2 \sin \gamma_2 + c_2 \cos \gamma_2) = 0, \end{aligned} \tag{3.5}$$

$$\begin{aligned} &4[(C_2 - B_2)/M + b_2^2 - c_2^2]\sin \gamma_2 \cos \gamma_2 + b_2 c_2 (\cos^2 \gamma_2 - \sin^2 \gamma_2) + \\ &-(b_2 \sin \gamma_2 + c_2 \cos \gamma_2)(b_1 \cos \gamma_1 - c_1 \sin \gamma_1) + \\ &+3\omega_0^2 (b_2 \cos \gamma_2 - c_2 \sin \gamma_2)(b_1 \sin \gamma_1 + c_1 \cos \gamma_1) = 0. \end{aligned}$$

Which allow us to determine the equilibrium orientation for the system of two bodies connected by a spherical hinge in the orbital coordinate system.

Equations (3.5) form a closed system of two equations with respect to the two aircraft angles. Trigonometric system (3.5) cannot be solved directly.

3.3 Investigations of Equilibria

For system (3.5) we used the universal change of sines and cosines through the tangent $t_i = \tan(\gamma_i)$

$$\sin \gamma_i = \frac{\tan(\gamma_i)}{\sqrt{1 + \tan^2(\gamma_i)}} = \frac{t_i}{\sqrt{1 + t_i^2}}, \quad \cos \gamma_i = \frac{1}{\sqrt{1 + \tan^2(\gamma_i)}} = \frac{1}{\sqrt{1 + t_i^2}}.$$

and obtain from (5) two equations with two unknowns t_1 and t_2

$$\begin{aligned} & 4\sqrt{(1 + t_1^2)(1 + t_2^2)} [((C_1 - B_1)/M + b_1^2 - c_1^2)t_1 + b_1c_1(1 - t_1^2)] = \\ & = (1 + t_1^2)[(b_1t_1 + c_1)(b_2 - c_2t_2) + 3(b_1 - c_1t_1)(b_2t_2 + c_2)], \\ & 4\sqrt{(1 + t_1^2)(1 + t_2^2)} [((C_2 - B_2)/M + b_2^2 - c_2^2)t_2 + b_2c_2(1 - t_2^2)] = \\ & = (1 + t_2^2)[(b_2t_2 + c_2)(b_1 - c_1t_1) + 3(b_2 - c_2t_2)(b_1t_1 + c_1)]. \end{aligned} \tag{3.6}$$

Then, we first divide and second multiply the left-hands and the right-hands sides of these equations and obtain two algebraic equations with two unknowns t_1 and t_2 .

3.3 Investigations of Equilibria

$$\begin{aligned} a_0 t_1^3 + a_1 t_1^2 + a_2 t_1 + a_3 &= 0, \\ b_0 t_1^2 + b_1 t_1 + b_2 &= 0. \end{aligned} \tag{3.7}$$

Where

$$\begin{aligned} a_0 &= (b_1 b_2 c_1^2 + 3b_1^2 c_1 c_2 - 3b_2^2 c_1 c_2 - b_1 b_2 c_2^2) t_2^3 + \\ &+ [3b_2 c_1 (b_2^2 - 2c_2^2 + d_2) - 3b_1^2 b_2 c_1 + b_1 c_2 (2b_2^2 + c_1^2 - c_2^2 + d_2)] t_2^2 + \\ &+ [3b_1^2 c_1 c_2 - b_1 b_2 (b_2^2 - c_1^2 - 2c_2^2 + d_2) + 3c_1 c_2 (2b_2^2 - c_2^2 + d_2)] t_2 + \\ &+ 3b_2 c_1 c_2^2 - 3b_1^2 b_2 c_1 + b_1 c_2 (c_1^2 - b_2^2), \\ a_1 &= [3b_1 c_2 (b_2^2 + 2c_1^2 - d_1) - 2b_1^2 b_2 c_1 - 3b_1^3 c_2 + b_2 c_1 (c_1^2 - c_2^2 - d_1)] t_2^3 + \\ &+ [3b_1^3 b_2 - 2b_1^2 c_1 c_2 - 3b_1 b_2 (b_1^2 + 2c_1^2 - 2c_2^2 - d_1 + d_2) + \\ &+ c_1 c_2 (2b_2^2 + c_1^2 - c_2^2 - d_1 + d_2)] t_2^2 - [2b_1^2 b_2 c_1 + 3b_1^3 c_2 + \\ &+ b_2 c_1 (b_2^2 - c_1^2 - 2c_2^2 + d_1 + d_2) + 3b_1 c_2 (2b_2^2 - 2c_1^2 - c_2^2 + d_1 + d_2)] t_2 + \\ &+ 3b_1^3 b_2 - 2b_1^2 c_1 c_2 + c_1 c_2 (c_1^2 - b_2^2 - d_1) - 3b_1 b_2 (2c_1^2 + c_2^2 - d_1), \end{aligned}$$

3.3 Investigations of Equilibria

$$\begin{aligned} a_0 t_1^3 + a_1 t_1^2 + a_2 t_1 + a_3 &= 0, \\ b_0 t_1^2 + b_1 t_1 + b_2 &= 0. \end{aligned} \tag{3.7}$$

Where

$$\begin{aligned} a_0 &= (b_1 b_2 c_1^2 + 3b_1^2 c_1 c_2 - 3b_2^2 c_1 c_2 - b_1 b_2 c_2^2) t_2^3 + \\ &+ [3b_2 c_1 (b_2^2 - 2c_2^2 + d_2) - 3b_1^2 b_2 c_1 + b_1 c_2 (2b_2^2 + c_1^2 - c_2^2 + d_2)] t_2^2 + \\ &+ [3b_1^2 c_1 c_2 - b_1 b_2 (b_2^2 - c_1^2 - 2c_2^2 + d_2) + 3c_1 c_2 (2b_2^2 - c_2^2 + d_2)] t_2 + \\ &+ 3b_2 c_1 c_2^2 - 3b_1^2 b_2 c_1 + b_1 c_2 (c_1^2 - b_2^2), \\ a_1 &= [3b_1 c_2 (b_2^2 + 2c_1^2 - d_1) - 2b_1^2 b_2 c_1 - 3b_1^3 c_2 + b_2 c_1 (c_1^2 - c_2^2 - d_1)] t_2^3 + \\ &+ [3b_1^3 b_2 - 2b_1^2 c_1 c_2 - 3b_1 b_2 (b_1^2 + 2c_1^2 - 2c_2^2 - d_1 + d_2) + \\ &+ c_1 c_2 (2b_2^2 + c_1^2 - c_2^2 - d_1 + d_2)] t_2^2 - [2b_1^2 b_2 c_1 + 3b_1^3 c_2 + \\ &+ b_2 c_1 (b_2^2 - c_1^2 - 2c_2^2 + d_1 + d_2) + 3b_1 c_2 (2b_2^2 - 2c_1^2 - c_2^2 + d_1 + d_2)] t_2 + \\ &+ 3b_1^3 b_2 - 2b_1^2 c_1 c_2 + c_1 c_2 (c_1^2 - b_2^2 - d_1) - 3b_1 b_2 (2c_1^2 + c_2^2 - d_1), \end{aligned}$$

3.3 Investigations of Equilibria

$$\begin{aligned} a_2 = & [b_1^3 b_2 - 3b_1^2 c_1 c_2 + 3c_1 c_2 (c_1^2 - b_2^2 - d_1) + b_1 b_2 (d_1 - 2c_1^2 - c_2^2)] t_2^3 + \\ & + [6b_1^2 b_2 c_1 + b_1^3 c_2 + 3b_2 c_1 (b_2^2 - c_1^2 - 2c_2^2 + d_1 + d_2) + \\ & + b_1 c_2 (2b_2^2 - 2c_1^2 - c_2^2 + d_1 + d_2)] t_2^2 + [b_1^3 b_2 - 6b_1^2 b_2 c_1 - \\ & - b_1 b_2 (b_2^2 + 2c_1^2 - 2c_2^2 - d_1 + d_2) + 3c_1 c_2 (2b_2^2 + c_1^2 - c_2^2 - d_1 + d_2)] t_2 + \quad (3.8) \\ & + b_1^2 b_2 c_1 + b_1^3 c_2 + b_1 c_2 (d_1 - b_2^2 - 2c_1^2), \\ a_3 = & [b_2 c_1 (b_1^2 - c_2^2) + 3b_1 (b_2^2 - c_1^2)] t_2^3 + \\ & + [c_1 c_2 (b_1^2 + 2b_2^2 - c_2^2 + d_2) - 3b_1 b_2 (b_2^2 - c_1^2 - 2c_2^2 + d_2)] t_2^2 + \\ & + [b_2 c_1 (b_1^2 - b_2^2 - c_1^2 + 2c_2^2 - d_2) - 3b_1 c_2 (2b_2^2 + c_1^2 - c_2^2 + d_2)] t_2 + \\ & + c_1 c_2 (b_1^2 - b_2^2) + 3b_1 b_2 (c_1^2 - c_2^2), \end{aligned}$$

3.3 Investigations of Equilibria

$$\begin{aligned} b_0 &= -3(b_2c_1 - b_1c_2)^2 t_2^2 + \\ &+ [6b_2c_2(b_1^2 - c_1^2) - 2b_1c_1(3b_2^2 - 3c_2^2 + 8d_2)]t_2 - 3(b_1b_2 - c_1c_2)^2, \\ b_1 &= [6b_1c_1(b_2^2 - c_2^2) - 6b_1^2b_2c_2 + 2b_2c_2(3c_1^2 + 8d_1)]t_2^2 + \\ &+ 2[5b_2^2(c_1^2 - b_1^2) + 5c_2^2(b_1^2 - c_1^2) + 12b_1b_2c_1c_2 + \\ &+ 8(b_1^2 - c_1^2 + d_1)(b_2^2 - c_2^2 + d_2)]t_2 + \\ &+ 6b_2c_2(b_1^2 - c_1^2) + 6b_1c_1(c_2^2 - b_2^2) + 16b_2c_2d_1, \\ b_2 &= -3(b_1b_2 + c_1c_2)^2 t_2^2 + \\ &+ [6b_2c_2(c_1^2 - b_1^2) + 2b_1c_1(3b_2^2 - 3c_2^2 + 8d_2)]t_2 - 3(b_2c_1 - b_1c_2)^2. \end{aligned} \tag{3.8}$$

Here

$$d_1 = \frac{(C_1 - B_1)}{M}, \quad d_2 = \frac{(C_2 - B_2)}{M}.$$

3.3 Investigations of Equilibria

Using the Resultant concept we eliminate the variable t_1 from Eq.(3.7). Expanding the determinant of resultant matrix of Eq.(3.7) with the help of *Mathematica* matrix function, we obtain the 12th order algebraic equation in t_2 . After factorization this equation has the form

$$P_1(t_2)P_2(t_2)P_3(t_2) = 0. \quad (3.9)$$

Here

$$P_1(t_2) = b_2c_2t_2^2 - (b_2^2 - c_2^2 + d_2)t_2 - b_2c_2,$$

$$P_2(t_2) = [b_1c_1(c_2^2 - 9b_2^2)d_1 + 3c_1^2b_2c_2(d_1 - c_1^2) - 3b_1^2b_2c_2(2c_1^2 + d_1) - 3b_1^4b_2c_2]t_2^2 + [3(b_2^2 - c_2^2)(b_1^4 + c_1^2(c_1^2 - d_1) + b_1^2(2c_1^2 + d_1) - 20b_1b_2c_1c_2d_1)]t_2 + 3b_1^4b_2c_2 + 3c_1^2b_2c_2(c_1^2 - d_1) + b_1c_1(b_2^2 - 9c_2^2)d_1 + 3b_1^2b_2c_2(2c_1^2 + d_1),$$

$$P_3(t_2) = p_0t_2^8 + p_1t_2^7 + p_2t_2^6 + p_3t_2^5 + p_4t_2^4 + p_5t_2^3 + p_6t_2^2 + p_7t_2 + p_8, \quad (3.10)$$

here p_i — are rather complicated coefficients, depending on 6 parameters. Using Eqs.(3.9) and (3.7), for each set of the system parameters, we can determine numerically the angles γ_2 and γ_1 , that is, all the equilibrium orientations of the satellite–stabilizer system.

3.3 Investigations of Equilibria

In studying the two-body system equilibrium orientations, we determine the domains with an equal number of real roots of Eq.(3.9) in the space of 6 parameters. The decomposition of the space of parameters into domains with an equal number of real roots is determined by the discriminant hypersurface. The form of the discriminant of the polynomial $P_3(t_2)$ is a very cumbersome expression.

Let us consider a simpler case when $b_1 = b_2 = c_1 = c_2 = b$.

In this case from system (3.10) we will obtain the equation of the 6th degree

$$P_1(t_2) = p_{00}t_2^6 + p_{01}t_2^5 + p_{02}t_2^4 + p_{03}t_2^3 + p_{04}t_2^2 + p_{05}t_2 + p_{06} = 0, \quad (3.11)$$

where

$$d_{01} = (A_1 - C_1)/Mb^2; \quad d_{02} = (A_2 - C_2)/Mb^2;$$

$$p_{00} = 12(1 + d_{01})(2d_{01}d_{02} - d_{01} - d_{02});$$

$$p_{01} = d_{01}^2(23 + 64d_{02} - 76d_{02}^2) + d_{02}(96 - 89d_{02}) + 6d_{01}(16 - 19d_{02}),$$

$$p_{02} = 4[d_{01}^2(16d_{02}^3 - 8d_{02}^2 - 58d_{02} + 23) + 4d_{02}(44d_{02}^2 - 32d_{02} - 51) + 3d_{01}(12d_{02}^2 - 5d_{02} - 17)],$$

3.3 Investigations of Equilibria

$$p_{03} = -2[d_{01}^2(8d_{02}^4 - 116d_{02}^2 + 105) + d_{02}^2(32d_{02}^2 - 231) + 6d_{01}d_{02}(4d_{02}^2 - 29)],$$

$$p_{04} = -4[d_{01}^2(16d_{02}^3 + 8d_{02}^2 - 58d_{02} - 23) + d_{02}(44d_{02}^2 + 32d_{02} - 51) + 3d_{01}(12d_{02}^2 + 5d_{02} - 17)],$$

$$p_{05} = d_{01}^2(23 - 64d_{02} - 76d_{02}^2) - d_{02}(96 + 89d_{02}) - 6d_{01}(16 + 19d_{02}),$$

$$p_{06} = -12(d_{01} - 1)(2d_{01}d_{02} + d_{01} + d_{02}).$$

Discriminant hypersurface of the polynomial (3.11) $P_1(t_2)$ (resultant of the two polynomials $P_1(t_2)$ and $P'_1(t_2)$) has the form

$$P_2(d_{01}, d_{02}) = \tag{3.12}$$

$$= (d_{01} + 1)(2d_{01}d_{02} - d_{01} - d_{02})P_3(d_{01}, d_{02})P_4(d_{01}, d_{02})P_5^2(d_{01}, d_{02})P_6(d_{01}, d_{02}) = 0.$$

Here

$$P_3(d_{01}, d_{02}) = (d_{01}^2d_{02}^2 + 4d_{01}^2 + 4d_{02}^2 - 2d_{01}d_{02} + 9),$$

$$P_4(d_{01}, d_{02}) = 4(d_{01} + d_{02})^2 + d_{01}^2d_{02}^2.$$

3.3 Investigations of Equilibria

$P_5(d_{01}, d_{02})$ and $P_6(d_{01}, d_{02})$ are the polynomials of the 12th and 16th degree.

Now we should check the change in the number of equilibria when the curve (3.12) is intersected. This can be done numerically by determining the number of equilibria at a single point of each domain at the plane (d_{01}, d_{02}) .

Only the curve $(2d_{01}d_{02} - d_{01} - d_{02})P_6(d_{01}, d_{02}) = 0$ separates the domains with different number of equilibria.

Fig.3.2 shows the distributions of domains with equal number of real roots of Eq.(3.11) and indicates the domains where 6 and 4 and 2 real solutions exist (12, 8 and 4 equilibrium orientations).

Therefore, in the case when $b_1 = b_2 = c_1 = c_2 = b$, there exist only 12, 8 and 4 equilibrium orientations for the satellite–stabilizer system.

3.3 Investigations of Equilibria

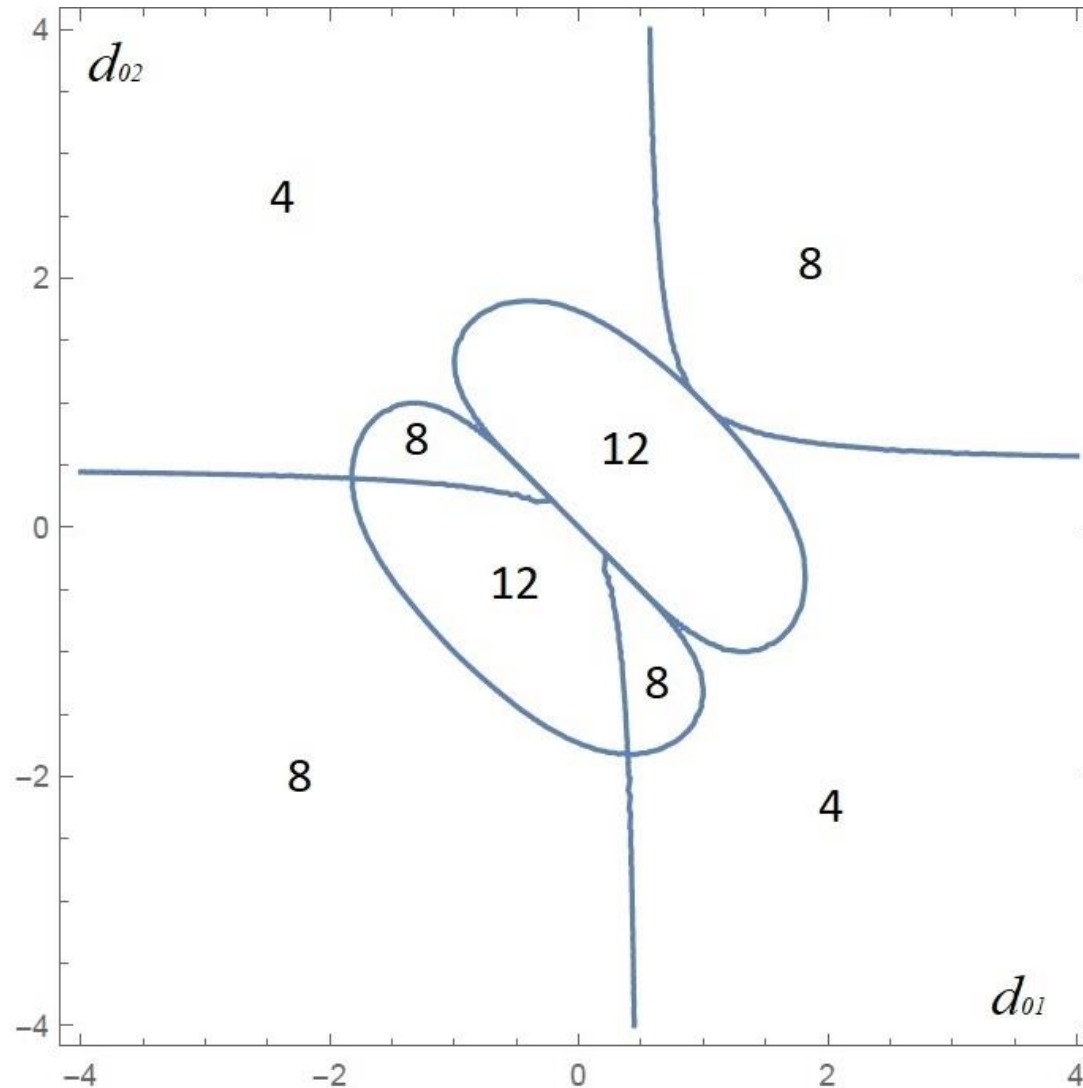


Fig. 3.2. The regions with the fixed number of equilibria

4. Spatial Equilibria

By assuming $\alpha_i = \text{const}$, $\beta_i = \text{const}$, $\gamma_i = \text{const}$ we obtain the stationary equations

$$\begin{aligned} a_{22}a_{23} - 3a_{32}a_{33} &= 0, \\ (a_{23}a_{21} - 3a_{33}a_{31}) + m_1(b_{21}a_{23} - 3b_{31}a_{33}) &= 0, \\ (a_{21}a_{22} - 3a_{31}a_{32}) - n_1(b_{21}a_{22} - 3b_{31}a_{32}) &= 0, \\ b_{22}b_{23} - 3b_{32}b_{33} &= 0, \\ (b_{23}b_{21} - 3b_{33}b_{31}) + m_2(a_{21}b_{23} - 3a_{31}b_{33}) &= 0, \\ (b_{21}b_{22} - 3b_{31}b_{32}) - n_2(a_{21}b_{22} - 3a_{31}b_{32}) &= 0, \\ a_{21}^2 + a_{22}^2 + a_{23}^2 - 1 &= 0, \quad b_{21}^2 + b_{22}^2 + b_{23}^2 - 1 = 0, \\ a_{31}^2 + a_{32}^2 + a_{33}^2 - 1 &= 0, \quad b_{31}^2 + b_{32}^2 + b_{33}^2 - 1 = 0, \\ a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} &= 0, \\ b_{21}b_{31} + b_{22}b_{32} + b_{23}b_{33} &= 0. \end{aligned} \tag{4.1}$$
$$m_1 = \frac{Ma_1a_2}{(A_1 - C_1) - Ma_1^2},$$
$$m_2 = \frac{Ma_1a_2}{(A_2 - C_2) - Ma_2^2},$$
$$n_1 = \frac{Ma_1a_2}{(B_1 - A_1) + Ma_1^2},$$
$$n_2 = \frac{Ma_1a_2}{(B_2 - A_2) + Ma_2^2}.$$

Which allow us to determine the equilibrium orientation for the system of two bodies connected by a spherical hinge in the case when $b_1=b_2=c_1=c_2=0$ in the orbital coordinate system. **See ref [4].**

4. Investigations of Spatial Equilibria

For the system of equations (4.1) we state the following problem: for given values of parameters it is required to determine all 12 directional cosines, i.e., all equilibrium orientations of a satellite-stabilizer in the orbital coordinate system. The other six direction cosines can be obtained from the orthogonality conditions.

We solved system (4.1) for the case, when $m_1 = m_2 = n_1 = n_2 = m$

To solve the algebraic system (4.1) we applied the algorithm of constructing the Groebner bases, using the **Grobner[gbasis]** Maple 18 package. The total time of the Groebner basis computation amounts to **24 h** of CPU time on a server with 16 Intel Xeon processors and 128 GB of RAM. The number of polynomials in the constructed Groebner basis was 160, the size of which occupies more than 1 million lines.

Later we constructed the Groebner bases for this case using **GroebnerBasis** package implemented in CAS Wolfram Mathematica 12.1 on PC with 16 GB of RAM and 2.90 GHz Intel Core i7-7500U processor in **31 hours**.

**GroebnerBasis[{f1,f2,f3,f4,f5,f6,f7,f8,f9,f10,f11,f12},
{b21,b22,b23,b31,b32,b33,a21,a22,a23,a31,a32,a33}]**

4. Investigations of Spatial Equilibria

Here we write down the polynomial in the Groebner basis that depends only on one variable $x = a_{33}$

$$P(x) = x(x^2 - 1)P_1(x)P_2(x)P_3(x)P_4(x)P_5(x)P_6(x)P_7(x)P_8(x) = 0, \quad (4.2)$$

where

$$P_1(x) = x^2 + m^2 - 1,$$

$$P_2(x) = 16x^4 + 8(m^2 - 2)x^2 + m^2,$$

$$P_3(x) = 16m^4(m^2 - 1)x^4 + 8m^2(2 + m^2 - 3m^4)x^2 + 9m^6 + 3m^4 - 8m^2 - 16,$$

$$P_4(x) = 12x^2 - m^3 - 4m^2 - 5m - 2,$$

$$P_5(x) = 12x^2 + m^3 - 4m^2 + 5m - 2,$$

$$P_6(x) = 144m^8x^4 - 24m^4(m^2 - 1)(7m^2 + 4)x^2 - \\ - 3m^{10} + 22m^8 - 63m^6 + 124m^4 - 144m^2 + 4,$$

$$P_7(x) = 144m^8(m^2 - 1)x^4 - 24m^4(m^2 - 1)(4m^4 + 3m^2 - 4)x^2 - \\ - (m^6 + m^4 - m^2 - 4)(3m^4 + 5m^2 + 4)(m^2 - 4),$$

$$P_8(x) = 16(m^2 + 1)(m^2 + 4)x^4 + 8(m^2 + 1)(m^2 + 4)(m^2 - 2)x^2 + m^2(m^2 - 2)^2.$$

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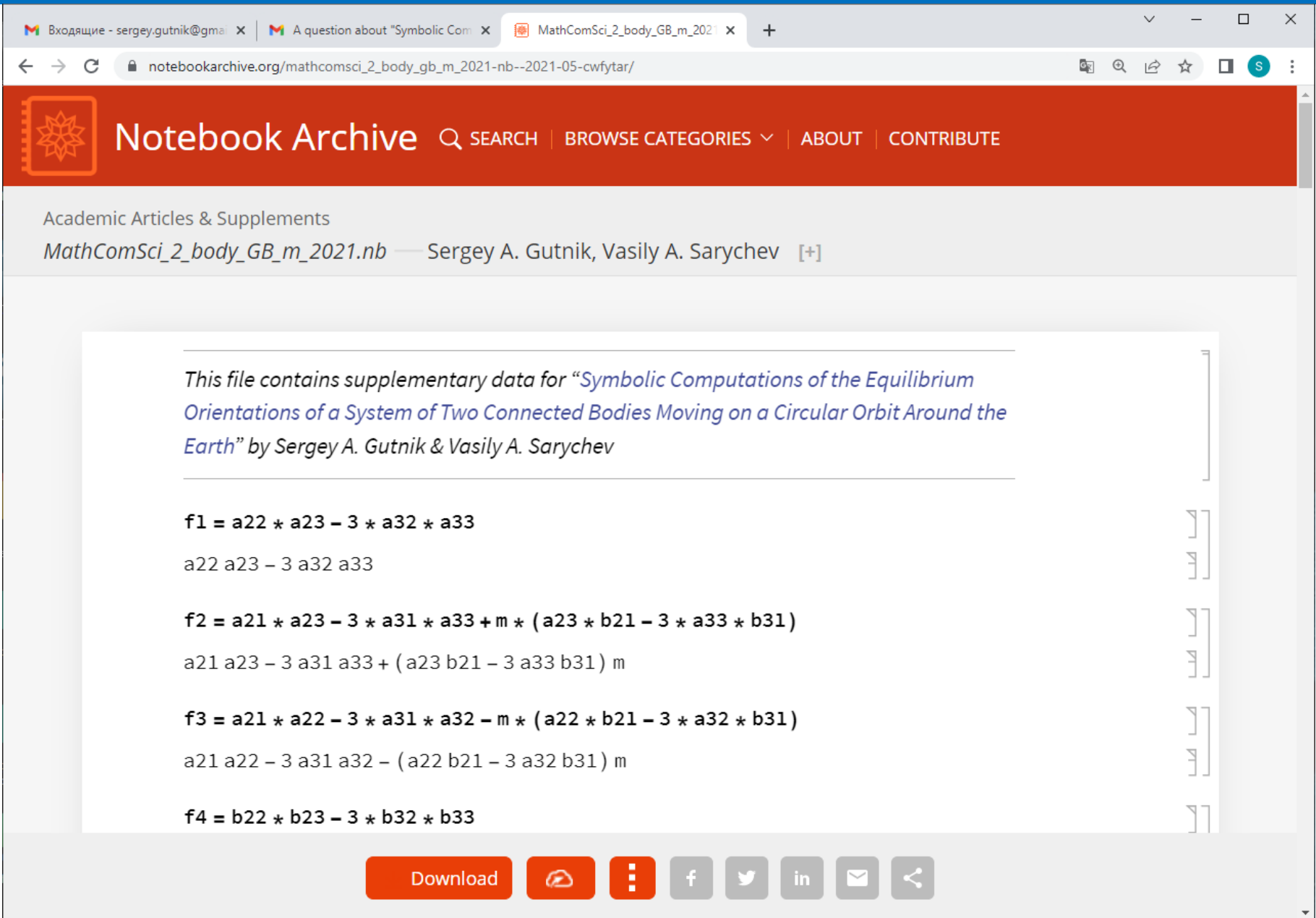
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This file contains supplementary data for “Symbolic Computations of the Equilibrium Orientations of a System of Two Connected Bodies Moving on a Circular Orbit Around the Earth” by Sergey A. Gutnik & Vasily A. Sarychev

$$f1 = a22 * a23 - 3 * a32 * a33$$

a22 a23 - 3 a32 a33

$$f2 = a21 * a23 - 3 * a31 * a33 + m * (a23 * b21 - 3 * a33 * b31)$$

a21 a23 - 3 a31 a33 + (a23 b21 - 3 a33 b31) m

$$f3 = a21 * a22 - 3 * a31 * a32 - m * (a22 * b21 - 3 * a32 * b31)$$

a21 a22 - 3 a31 a32 - (a22 b21 - 3 a32 b31) m

$$f4 = b22 * b23 - 3 * b32 * b33$$

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$$\begin{aligned} &790\,430\,220\,288\,a^{33^{23}}m^{38} + 204\,352\,782\,336\,a^{33^{25}}m^{38} - 122\,305\,904\,640\,a^{33^{27}}m^{38} + 12\,230\,590\,464\,a^{33^{29}}m^{38} - 767\,708\,a^{33}m^{40} + \\ &11\,074\,016\,a^{33^3}m^{40} + 33\,154\,996\,a^{33^5}m^{40} - 1\,850\,353\,576\,a^{33^7}m^{40} + 19\,974\,604\,528\,a^{33^9}m^{40} - 112\,023\,657\,984\,a^{33^{11}}m^{40} + \\ &344\,564\,769\,792\,a^{33^{13}}m^{40} - 567\,595\,155\,456\,a^{33^{15}}m^{40} + 379\,248\,816\,128\,a^{33^{17}}m^{40} + 245\,739\,749\,376\,a^{33^{19}}m^{40} - 671\,650\,873\,344\,a^{33^{21}}m^{40} + \\ &524\,669\,681\,664\,a^{33^{23}}m^{40} - 185\,582\,223\,360\,a^{33^{25}}m^{40} + 24\,461\,180\,928\,a^{33^{27}}m^{40} + 360\,058\,a^{33}m^{42} - 4\,992\,904\,a^{33^3}m^{42} - \\ &2\,332\,802\,a^{33^5}m^{42} + 520\,994\,528\,a^{33^7}m^{42} - 4\,563\,880\,656\,a^{33^9}m^{42} + 13\,778\,373\,248\,a^{33^{11}}m^{42} - 2\,092\,940\,544\,a^{33^{13}}m^{42} - \\ &87\,642\,159\,104\,a^{33^{15}}m^{42} + 243\,130\,368\,000\,a^{33^{17}}m^{42} - 323\,158\,376\,448\,a^{33^{19}}m^{42} + 238\,429\,274\,112\,a^{33^{21}}m^{42} - 93\,682\,925\,568\,a^{33^{23}}m^{42} + \\ &15\,288\,238\,080\,a^{33^{25}}m^{42} - 105\,609\,a^{33}m^{44} + 1\,146\,068\,a^{33^3}m^{44} + 4\,593\,325\,a^{33^5}m^{44} - 118\,430\,968\,a^{33^7}m^{44} + 421\,764\,352\,a^{33^9}m^{44} + \\ &696\,353\,792\,a^{33^{11}}m^{44} - 8\,239\,346\,432\,a^{33^{13}}m^{44} + 24\,126\,953\,472\,a^{33^{15}}m^{44} - 36\,916\,273\,152\,a^{33^{17}}m^{44} + 32\,232\,701\,952\,a^{33^{19}}m^{44} - \\ &15\,267\,004\,416\,a^{33^{21}}m^{44} + 3\,057\,647\,616\,a^{33^{23}}m^{44} + 18\,900\,a^{33}m^{46} - 99\,189\,a^{33^3}m^{46} - 1\,697\,927\,a^{33^5}m^{46} + 11\,195\,784\,a^{33^7}m^{46} + \\ &17\,707\,296\,a^{33^9}m^{46} - 252\,999\,040\,a^{33^{11}}m^{46} + 706\,875\,648\,a^{33^{13}}m^{46} - 918\,503\,424\,a^{33^{15}}m^{46} + 586\,137\,600\,a^{33^{17}}m^{46} - \\ &148\,635\,648\,a^{33^{19}}m^{46} - 1890\,a^{33}m^{48} - 4941\,a^{33^3}m^{48} + 165\,495\,a^{33^5}m^{48} + 153\,864\,a^{33^7}m^{48} - 3\,124\,944\,a^{33^9}m^{48} + 7\,319\,040\,a^{33^{11}}m^{48} - \\ &6\,718\,464\,a^{33^{13}}m^{48} + 2\,211\,840\,a^{33^{15}}m^{48} + 81\,a^{33}m^{50} + 999\,a^{33^3}m^{50} + 792\,a^{33^5}m^{50} - 13\,392\,a^{33^7}m^{50} + 20\,736\,a^{33^9}m^{50} - 9216\,a^{33^{11}}m^{50} \end{aligned}$$

Factor[G2]

$$\begin{aligned} &(-1 + a^{33}) a^{33} (1 + a^{33}) (-2 + m) (-1 + m) (1 + m) (2 + m) (1 + m^2) (-1 + a^{33^2} + m^2) (-16 a^{33^2} + 16 \\ &a^{33^4} + m^2 + 8 a^{33^2} m^2) (-2 - 5 m - 4 m^2 + 12 a^{33^2} m^2 - m^3) (-2 + 5 m - 4 m^2 + 12 a^{33^2} m^2 + m^3) (-64 a^{33^2} \\ &+ 64 a^{33^4} + 4 m^2 - 48 a^{33^2} m^2 + 80 a^{33^4} m^2 - 4 m^4 + 24 a^{33^2} m^4 + 16 a^{33^4} m^4 + m^6 + 8 a^{33^2} m^6) (-16 - 8 m^2 \\ &+ 16 a^{33^2} m^2 + 3 m^4 + 8 a^{33^2} m^4 - 16 a^{33^4} m^4 + 9 m^6 - 24 a^{33^2} m^6 + 16 a^{33^4} m^6) (64 - 144 m^2 + 124 m^4 + 96 \\ &a^{33^2} m^4 - 63 m^6 + 72 a^{33^2} m^6 + 22 m^8 - 168 a^{33^2} m^8 + 144 a^{33^4} m^8 - 3 m^{10}) (-64 - 80 m^2 - 28 m^4 - 96 a^{33^2} \\ &m^4 + 37 m^6 + 168 a^{33^2} m^6 + 26 m^8 + 24 a^{33^2} m^8 - 144 a^{33^4} m^8 + 4 m^{10} - 96 a^{33^2} m^{10} + 144 a^{33^4} m^{10} - 3 m^{12}) \end{aligned}$$

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6. Conclusion

- We have obtained all equilibrium orientations of satellite–stabilizer system in the plane of a circular orbit and in the plane perpendicular and tangent to the orbital reference frame, using computer algebra method (based on the resultant calculations)
- The conditions for the existence of these equilibria were obtained
- An analysis of the evolution of domains of existence of equilibrium orientations in the plane of system parameters d_{01} and d_{02} for the special case when the coordinates of the spherical hinge in the satellite body coordinate system $Ox_1y_1z_1$ and stabilizer body coordinate system $Ox_2y_2z_2$ are equal have made using discriminant hypersurface approach
- To find equilibrium orientations, the polynomial algebra methods were used

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