Painleve Test and Integrability of Polynomial ODEs

A.B. Aranson
aboar@yandex.ru

quit Scientific Research Institute of Long-Range Radio Communication
Moscow, Russia

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Lotka-Volterra Equations

V.I. Arnold, Ordinary Differential Equations, Springer-Verlag, 1992.

$$\frac{dx}{dt} = kx - axy,$$
$$\frac{dy}{dt} = -ly + bxy,$$

where t - independent variable, x, y - dependent variables, $k, a, l, b > 0$ - parameters.
The first integral is:

$$y^k e^{-ay} x^l e^{-bx} = C$$

Power Series Substitution

We find solutions of Lotka-Volterra system in form of Puiseux series with finite nonzero principal part

$$x(t) = t^{\alpha_1}(x_0 + \sum_{j=1}^{\infty} x_j t^{j\Delta}),$$
$$y(t) = t^{\alpha_2}(y_0 + \sum_{j=1}^{\infty} y_j t^{j\Delta}),$$

where coefficients $x_0, y_0 \neq 0$, powers $\alpha_1, \alpha_2 < 0$ - rational, $\Delta > 0$ - rational.

After substitution of series to source equations they are transformed to series

$$t^{\beta_1}(c_{1,0} + \sum_{j=0}^{\infty} c_{1,j} t^{j\Delta}) = 0,$$
$$t^{\beta_2}(c_{2,0} + \sum_{j=0}^{\infty} c_{2,j} t^{j\Delta}) = 0,$$

coefficients x_0, x_j, y_0, y_j are solutions of equations $c_{1,0}(x_0, y_0) = 0$, $c_{1,j}(x_j, y_j) = 0$, $c_{2,0}(x_0, y_0) = 0$, $c_{2,j}(x_j, y_j) = 0$. Powers β_1, β_2 we consider later.

Powers and Coefficients of minimal power Terms

At first we substitute terms $x_0 t^{\alpha_1}, y_0 t^{\alpha_2}$ to Lotka-Volterra system and every monomial of source equations is transformed to power function. Power of that function is power of the source monomial.

$$-\alpha_1 x_0 t^{\alpha_1 - 1} + k x_0 t^{\alpha_1} - a x_0 y_0 t^{\alpha_1 + \alpha_2},$$
$$-\alpha_2 y_0 t^{\alpha_2 - 1} - l y_0 t^{\alpha_2} + b x_0 y_0 t^{\alpha_1 + \alpha_2},$$

Different sets of power functions with minimal power $\beta_1 = \min(\alpha_1 - 1, \alpha_1, \alpha_1 + \alpha_2)$, $\beta_2 = \min(\alpha_2 - 1, \alpha_2, \alpha_1 + \alpha_2)$ corresponds to different α_1, α_2 values.

Let consider power β_1. Because $\alpha_1 - 1 < \alpha_1$ we have only 3 cases

$\beta_1 = \alpha_1 - 1 < \alpha_1 + \alpha_2$,

$\beta_1 = \alpha_1 + \alpha_2 < \alpha_1 - 1$,

$\beta_1 = \alpha_1 - 1 = \alpha_1 + \alpha_2$.

Powers and Coefficients of minimal power Terms

$\beta_1 = \alpha_1 - 1 < \alpha_1 + \alpha_2$, then function $-\alpha_1 x_0 t^{\alpha_1 - 1}$ has minimal power and coefficient $c_{1,0} = -\alpha_1 x_0 = 0$ if $\alpha_1 = 0$ or $x_0 = 0$ that contrary to conditions $\alpha_1 < 0$, $x_0 \neq 0$.
$\beta_1 = \alpha_1 + \alpha_2 < \alpha_1 - 1$, then function $-a x_0 y_0 t^{\alpha_1 + \alpha_2}$ has minimal power and coefficient $c_{1,0} = -a x_0 y_0 = 0$ if $a = 0$ or $x_0 = 0$ or $y_0 = 0$ that contrary to conditions $a > 0$, $x_0, y_0 \neq 0$.
$\beta_1 = \alpha_1 - 1 = \alpha_1 + \alpha_2$, then $\alpha_2 = -1$ and two functions $-\alpha_1 x_0 t^{\alpha_1 - 1}$, $-a x_0 y_0 t^{\alpha_1 - 1}$ have minimal power and coefficient
$c_{1,0} = -\alpha_1 x_0 - a x_0 y_0 = 0$ then $y_0 = -\alpha_1 / a$.
Similarly $\beta_2 = \alpha_2 - 1 = \alpha_1 + \alpha_2$, then $\alpha_1 = -1$ and
$c_{2,0} = -\alpha_2 y_0 + b x_0 y_0 = 0$ then $x_0 = \alpha_2 / b$.
Finally $\alpha_1 = \alpha_2 = -1$, $\beta_1 = \beta_2 = -2$, $x_0 = -1/b$, $y_0 = 1/a$

Next Terms

To calculate next terms we substitute to source equations first two terms of the solution expansion with already calculated powers and coefficients

$$x = -t^{-1}/b + x_1 t^{-1+\Delta},$$
$$y = t^{-1}/a + y_1 t^{-1+\Delta},$$

and reduce similar terms. For first equation of Lotka-Volterra system we have

$$(ay_1/b - x_1\Delta)t^{-2+\Delta} - kt^{-1}/b + kx_1 t^{-1+\Delta} - ax_1 y_1 t^{-2+2\Delta},$$

Powers $-2+\Delta < -2+2\Delta$ and $-1 < -1+\Delta$, so we consider terms with powers $-2+\Delta$ and -1 only.
Similarly, substitution and reduction for second equation is

$$(bx_1/a - y_1\Delta)t^{-2+\Delta} - lt^{-1}/a,$$

Next Terms

If $0 < \Delta < 1$ then coefficients x_1, y_1 are solutions of the homogenious linear algebraic equations system

$$-\Delta x_1 + (a/b)y_1 = 0,$$
$$(b/a)x_1 - \Delta y_1 = 0,$$

but this system doesn't have solutions if $\Delta \neq \pm 1$.
If $\Delta = 1$ then coefficients x_1, y_1 are solutions of the linear algebraic equations system

$$-x_1 + (a/b)y_1 - k/b = 0,$$
$$(b/a)x_1 - y_1 - l/a = 0,$$

but this system has solution if $k = -l$ only that contradict to condition $k, l > 0$.
Lotka-Volterra system doesn't pass Painleve test with allowed values of parameters.

Euler-Poisson Equations

V.V. Golubev, Lectures on Integration of Equations Motion of a Heavy Rigid Body near Fixed Point, Moscow: GITTL, 1953.

$$A\frac{dp}{dt} = (B - C)qr - Mg(z_0\gamma_2 - y_0\gamma_3),$$
$$B\frac{dq}{dt} = (C - A)rp - Mg(x_0\gamma_3 - z_0\gamma_1),$$
$$C\frac{dr}{dt} = (A - B)pq - Mg(y_0\gamma_1 - x_0\gamma_2),$$
$$\frac{d\gamma_1}{dt} = r\gamma_2 - q\gamma_3,$$
$$\frac{d\gamma_2}{dt} = p\gamma_3 - r\gamma_1,$$
$$\frac{d\gamma_3}{dt} = q\gamma_1 - p\gamma_2,$$

Parameters of Euler-Poisson Equations

where t - time, A, B, C - principal moments of inertia, which satisfy triangle inequalities

$$A > 0,\ B > 0,\ C > 0,$$
$$A + B \geq C,\ A + C \geq B,\ B + C \geq A,$$

Mg - the body weight, x_0, y_0, z_0 - coordinates of the center of gravity of the rigid body in the body frame, p, q, r - projections of the angular velocity vector onto the body frame axes, $\gamma_1, \gamma_2, \gamma_3$ - direction cosines of the vertical in the body frame.

First Integrals of Euler-Poisson Equations

$$
\begin{aligned}
&Ap^2 + Bq^2 + Cr^2 - 2Mg(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3) = h = \text{const},\\
&Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 = l = \text{const},\\
&\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1.
\end{aligned}
$$

These are energy, momentum and geometry integrals.

We take a system of units where $Mg = 1$.

Known Solutions

$x_0 = y_0 = z_0 = 0$ Euler,
$x_0 = y_0 = 0$, $A = B$ Lagrange,
$y_0 = z_0 = 0$, $A = B = 2C$ S.Kowalevski,
$y_0 = z_0 = 0$, $A = 2C$, $A < B < 3A$ Bobylev-Steklov,
$y_0 = z_0 = 0$, $(A - 2B)(A - 2C) < 0$ Steklov,
$y_0 = z_0 = 0$, $A = 16C(C - B)/(8C - 9B)$ Goryachev,
$y_0 = z_0 = 0$, $B = 4A(2C - A)/(17C - 8A)$ Konosevich-Pozdnyakovich,
$y_0 = z_0 = 0$, $A = 18C(C - B)/(9C - 10B)$ N.Kowalevski, Dokshevich,
$y_0 = z_0 = 0$, $C = 9A(2B - A)/(2(16B - 9A))$ Chaplygin,
$y_0 = z_0 = 0$, $A = B = 4C$ Goryachev-Chaplygin,
$y_0 = 0$, $x_0\sqrt{A(B - C)} = z_0\sqrt{C(A - B)}$, $A > B > C$ Hess-Appelrot, Dokchevich,
$z_0 = 0$, $p(t) = q(t) = \gamma_3(t) = 0$ Mlodzievskii,
$y_0 = 0$, $x_0\sqrt{B - C} = z_0\sqrt{A - B}$, $A > B > C$ Grioly

I.N. Gashenenko, G.V. Gorr, A.M. Kovalev, Classical Problems of the Dynamics of a Rigid Body, Kiev, Naukova Dumka, 2012.

Power Substitution

We find solutions of Euler-Poisson system in form of Puiseux series with finite nonzero principal part

$$
\begin{aligned}
p(t) &= t^{\alpha_1}(p_0 + \textstyle\sum_{j=1}^{\infty} p_j t^{j\Delta}),\\
q(t) &= t^{\alpha_2}(q_0 + \textstyle\sum_{j=1}^{\infty} q_j t^{j\Delta}),\\
r(t) &= t^{\alpha_3}(r_0 + \textstyle\sum_{j=1}^{\infty} r_j t^{j\Delta}),\\
\gamma_1(t) &= t^{\alpha_4}(\gamma_{1,0} + \textstyle\sum_{j=1}^{\infty} \gamma_{1,j} t^{j\Delta}),\\
\gamma_2(t) &= t^{\alpha_5}(\gamma_{2,0} + \textstyle\sum_{j=1}^{\infty} \gamma_{2,j} t^{j\Delta}),\\
\gamma_3(t) &= t^{\alpha_6}(\gamma_{3,0} + \textstyle\sum_{j=1}^{\infty} \gamma_{3,j} t^{j\Delta}),
\end{aligned}
$$

where coefficients $p_0, q_0, r_0, \gamma_{1,0}, \gamma_{2,0}, \gamma_{3,0} \neq 0$, powers $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 < 0$ - rational, $\Delta > 0$ - rational.

Powers and Coefficients of minimal power Terms

Ssubstitute terms $p_0 t^{\alpha_1}$, $q_0 t^{\alpha_2}$, $r_0 t^{\alpha_3}$, $\gamma_{1,0} t^{\alpha_4}$, $\gamma_{2,0} t^{\alpha_5}$, $\gamma_{3,0} t^{\alpha_6}$ to Euler-Poisson system

$$
\begin{gathered}
-A\alpha_1 p_0 t^{\alpha_1-1} + (B-C) q_0 r_0 t^{\alpha_2+\alpha_3} + y_0\gamma_{3,0} t^{\alpha_6} - z_0\gamma_{2,0} t^{\alpha_5},\\
-B\alpha_2 q_0 t^{\alpha_2-1} + (C-A) p_0 r_0 t^{\alpha_1+\alpha_3} + z_0\gamma_{1,0} t^{\alpha_4} - x_0\gamma_{3,0} t^{\alpha_6},\\
-C\alpha_3 r_0 t^{\alpha_3-1} + (A-B) p_0 q_0 t^{\alpha_1+\alpha_2} + x_0\gamma_{2,0} t^{\alpha_5} - y_0\gamma_{1,0} t^{\alpha_4},\\
-\alpha_4\gamma_{1,0} t^{\alpha_4-1} + r_0\gamma_{2,0} t^{\alpha_3+\alpha_5} - q_0\gamma_{3,0} t^{\alpha_2+\alpha_6},\\
-\alpha_5\gamma_{2,0} t^{\alpha_5-1} + p_0\gamma_{3,0} t^{\alpha_1+\alpha_6} - r_0\gamma_{1,0} t^{\alpha_3+\alpha_4},\\
-\alpha_6\gamma_{3,0} t^{\alpha_6-1} + q_0\gamma_{1,0} t^{\alpha_2+\alpha_4} - p_0\gamma_{2,0} t^{\alpha_1+\alpha_5},\\
Ap_0^2 t^{2\alpha_1} + Bq_0^2 t^{2\alpha_2} + Cr_0^2 t^{2\alpha_3} - 2x_0\gamma_{1,0} t^{\alpha_4} - 2y_0\gamma_{2,0} t^{\alpha_5} - 2z_0\gamma_{3,0} t^{\alpha_6} - ht^0\\
Ap_0\gamma_{1,0} t^{\alpha_1+\alpha_4} + Bq_0\gamma_{2,0} t^{\alpha_2+\alpha_5} + Cr_0\gamma_{3,0} t^{\alpha_3+\alpha_6} - It^0,\\
\gamma_{1,0}^2 t^{2\alpha_4} + \gamma_{2,0}^2 t^{2\alpha_5} + \gamma_{3,0}^2 t^{2\alpha_6} - 1t^0.
\end{gathered}
$$

Powers and Coefficients of minimal power Terms

$$\beta_1 = \min(\alpha_1 - 1, \alpha_2 + \alpha_3, \alpha_6, \alpha_5),$$
$$\beta_2 = \min(\alpha_2 - 1, \alpha_1 + \alpha_3, \alpha_4, \alpha_6),$$
$$\beta_3 = \min(\alpha_3 - 1, \alpha_1 + \alpha_2, \alpha_5, \alpha_4),$$
$$\beta_4 = \min(\alpha_4 - 1, \alpha_3 + \alpha_5, \alpha_2 + \alpha_6),$$
$$\beta_5 = \min(\alpha_5 - 1, \alpha_1 + \alpha_6, \alpha_3 + \alpha_4),$$
$$\beta_6 = \min(\alpha_6 - 1, \alpha_2 + \alpha_4, \alpha_1 + \alpha_5),$$
$$\beta_7 = \min(2\alpha_1, 2\alpha_2, 2\alpha_3, \alpha_4, \alpha_5, \alpha_6, 0),$$
$$\beta_8 = \min(\alpha_1 + \alpha_4, \alpha_2 + \alpha_5, \alpha_3 + \alpha_6, 0),$$
$$\beta_9 = \min(2\alpha_4, 2\alpha_5, 2\alpha_6, 0),$$

To solve these equations we used author algorithm and author computer programm described in
A.B. Aranson, Computation of Collections of Correlate Faces for Several Polyhedrons, in Computer Algebra in Scientific Computing, Munchen: Techn, Univ. Munchen, 2003, pp. 13–17.

$z_0 = 0,\ x_0, y_0 \neq 0,\ B = A,$
$\alpha_1 = \alpha_2 = \alpha_3 = -1,\ \alpha_4 = \alpha_5 = -2,\ \alpha_6 = -2 + \eta,\ \eta > 0$
$\beta_1 = \beta_2 = \beta_3 = -2,\ \beta_4 = \beta_5 = \beta_6 = -3,\ \beta_7 = -2,\ \beta_8 = -3,\ \beta_9 = -4$

$$\begin{gathered} Ap_0 + (A - C)q_0 r_0 = 0, \\ Aq_0 + (C - A)p_0 r_0 = 0, \\ Cr_0 + x_0\gamma_{2,0} - y_0\gamma_{1,0} = 0, \\ 2\gamma_{1,0} + r_0\gamma_{2,0} = 0, \\ 2\gamma_{2,0} - r_0\gamma_{1,0} = 0, \\ q_0\gamma_{1,0} - p_0\gamma_{2,0} = 0, \\ Ap_0^2 + Bq_0^2 + Cr_0^2 - 2x_0\gamma_{1,0} - 2y_0\gamma_{2,0} = 0, \\ Ap_0\gamma_{1,0} + Bq_0\gamma_{2,0} = 0, \\ \gamma_{1,0}^2 + \gamma_{2,0}^2 = 0. \end{gathered}$$

Solution
$A = 2C,\ r - 0 = 2i,\ q_0 = p_0 i,\ \gamma_{1,0} - 2C/(x_0 + y_0 i),\ \gamma_{2,0} - 2Ci/(x_0 + y_0 i)$

$y_0 = z_0 = 0$, $x_0 \neq 0$, $B = A$,
$\alpha_1 = \alpha_2 = -1 - \eta_1$, $\alpha_3 = -1$, $\alpha_4 = \alpha_5 = -2$, $\alpha_6 = -2 + \eta_1 + \eta_2$,
$\eta_1, \eta_2 > 0$
$\beta_1 = \beta_2 = -2 - \eta_1$, $\beta_3 = -2$, $\beta_4 = \beta_5 = -3$, $\beta_6 = -3 - \eta_1$,
$\beta_7 = -2(1 + -\eta_1)$, $\beta_8 = -3 - \eta_1$, $\beta_9 = -4$

$$
\begin{gathered}
Ap_0 + (A - C)q_0 r_0 = 0, \\
Aq_0 + (C - A)p_0 r_0 = 0, \\
Cr_0 + x_0\gamma_{2,0} = 0, \\
2\gamma_{1,0} + r_0\gamma_{2,0} = 0, \\
2\gamma_{2,0} - r_0\gamma_{1,0} = 0, \\
q_0\gamma_{1,0} - p_0\gamma_{2,0} = 0, \\
Ap_0^2 + Bq_0^2 = 0, \\
Ap_0\gamma_{1,0} + Bq_0\gamma_{2,0} = 0, \\
\gamma_{1,0}^2 + \gamma_{2,0}^2 = 0.
\end{gathered}
$$

Solution $A = 2C/(1 - \eta_1)$, $r - 0 = 2i$, $q_0 = p_0 i$, $\gamma_{1,0} - 2C/x_0$, $\gamma_{2,0} - 2Ci/x_0$, $\eta_1 = 1/2$, $\eta_2 = 1$

New Cases of Expansibility

$z_0 = 0,\ x_0, y_0 \neq 0,\ B = A$
$z_0 = 0,\ x_0, y_0 \neq 0,\ C = B$

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Many thanks for your attention