The phase portrait of the pendulum model of resonance in (a, λ) coordinates

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Abstract. The model of pendulum is widely used to study resonance in celestial mechanics. Usually it is considered in $(\lambda, \dot{\lambda})$ coordinates where it is symmetric. However, when we rebuild it in (λ, a) coordinates, where λ is resonance argument, a is semimaajor axis the symmetry is loses. The migration rate (and rate of the resonance approach) is expressed in terms of semimajor axis. For example, Yarkovsky effect generates a linear drift in semimajor axis. Moreover, if we accept adiabatic changes of orbit when approach to resonance we can locally assume linear dependence semimajor axis on time in a very common case. Therefore this representation has some advantages.

Introduction

In detail, the pendulum model of resonance is described in [1]. The Hamiltonian of the three body problem in case of resonance can be reduced to the form:

$$H = 1/2\alpha\Lambda^2 + \mu A \cos\lambda \tag{1}$$

where:

$$\beta = \frac{3}{2k^2}\left[\frac{(j-k)^2}{ma^2} + \frac{j^2}{m_p a_p^2}\right] \tag{2}$$

$$\varepsilon(n^2)^{1-k/4} f_d \frac{a^{3-k}}{a_p}\frac{m_p}{m_c} m^{1-k/2} \tag{3}$$

$$m = \frac{m_p m_c}{m_p + m_c} \tag{4}$$

Here f_d - the according resonance term of the perturbation function expansion, m_c - central mass, m, a, n – mass, semimajor axis and mean motion of the asteroid,m_p, a_p, n_p – mass, semimajor axis and mean motion of the perturbing planet, j, k – integer, $\dot{\varpi}$ - secular perihelion precession rate of the asteroid.

The value of α is the measure of the neighborhood to the exact resonance.

1. The case of constant velocity

Usually, the phase portrait of the pendulum is drawn in $(\lambda, \dot{\lambda})$ coordinates, where it is obviously symmetric. However in the study of dynamic in the central gravity field it is natural to use other coordinates: radius vector r (or semimajor axis a) and longitude. In the model of resonance it is possible to identify λ with longitude. Therefore it is naturally to redraw the phase portrait in a, λ coordinates where it becomes asymmetric, Figure 1.

The exact solution of the pendulum equation is not suitable, because it contained the elliptic integral. By this reason, we consider asymptotic solution[2]. After the substitution $\dot{\lambda} = n - n_{res} = \delta_n$ for the circulation solution we have:

$$\dot{\lambda} = \Omega + \frac{\Omega}{\Omega^2} \cos \Omega t = \delta_n \pm \frac{1}{\delta_n} \cos \delta_n t \tag{5}$$

Denote $a - a_{res} = \delta_a$ and using the expansion:

$$\delta_n = \pm 3/2 \delta_a \frac{n_{res}}{a_{res}} + \frac{15}{8} \delta_a^2 \frac{n_{res}}{a_{res}^2} + ... \tag{6}$$

Finally we obtain:

$$\dot{\lambda} = a_{res}^{-3/2} \pm 3/2 \delta_a \frac{n_{res}}{a_{res}} + \frac{15}{8} \delta_a^2 \frac{n_{res}}{a_{res}^2} + ... \tag{7}$$

This result allows us to rebuild the phase portrait in $(a, \lambda\ )$ coordinates (fig.1). Evidently, the curves with $\dot{\lambda}$ which are symmetric in $(\dot{\lambda}, \lambda)$ coordinates are not symmetric in (a, λ) coordinates.

This representation has some advantages. The migration rate (and rate of the resonance approach) is expressed in terms of semimajor axis. For example, Yarkovsky effect generates a linear drift in semimajor axis.

As it is followed from expression above, λ_{res} is approximately proportional time. Therefore we can replace the horizontal x-axis (λ axis) in the phase portrait by the time axis. As consequence, the linear drift in the semimajor axis can be expressed as a tangent (slope) of the trajectory. On our opinion, such expression can be useful in the process of the resonance capture modelling.

Moreover, if we accept adiabatic changes of orbit when approach to resonance we can locally assume linear dependence semimajor axis on time in a very common case.

Conclusion

The pendulum model of the mean motion resonance is considered in (λ, a) coordinates where it is not symmetric. This representation has some advantages. The migration rate (and rate of the resonance approach) is expressed in terms of semimajor axis. For example, Yarkovsky effect generates a linear drift in semimajor axis. Our method allow to simple numerical modeling of the process of approach of the small body to resonance.

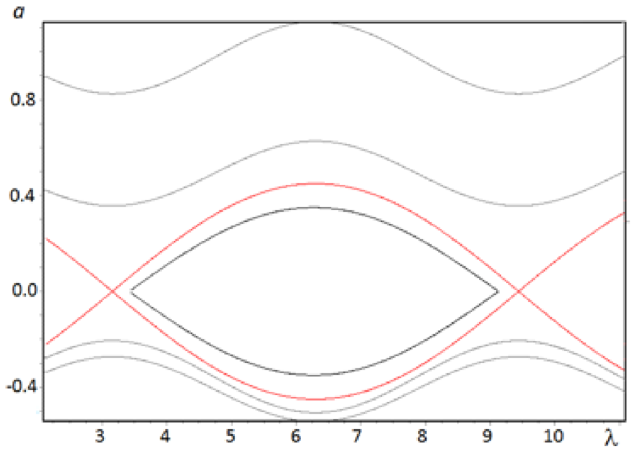

FIGURE 1. The phase portrait of asymmetric pendulum in a, λ coordinates

References

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[2] Moiseev N. N., Asymptotic Methods in Nonlinear Mechanics [in Russian], Moscow, Nauka 1981.

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