The phase portrait of the pendulum model of resonance in (a, λ) coordinates

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Introduction

The three body Hamiltonian in case of resonance can be reduced to the form:

$$H = 1/2\alpha\Lambda^2 + \mu A\cos\lambda \tag{1}$$

where:

$$\beta = \frac{3}{2k^2}\left[\frac{(j-k)^2}{ma^2} + \frac{j^2}{m_p a_p^2}\right] \tag{2}$$

$$\varepsilon(n^2)^{1-k/4} f_d \frac{a^{3-k}}{a_p}\frac{m_p}{m_c} m^{1-k/2} \tag{3}$$

$$m = \frac{m_p m_c}{m_p + m_c} \tag{4}$$

Here f_d - the according resonance term of the perturbation function expansion, m_c - central mass, m, a, n – mass, semimajor axis and mean motion of the asteroid,m_p, a_p, n_p – mass, semimajor axis and mean motion of the perturbing planet, j, k – integer, $\dot{\varpi}$ - secular perihelion precession rate of the asteroid.

To measure the distance to the exact resonance we use the parameter α' following Murray and Dermott (1999)

$$\alpha' = [(j-k)n - jn_p + k\dot{\varpi}], \tag{5}$$

The solution of pendulum equation in elliptic functions for case of circulation (Wisdom,1985) is:

$$\varphi = \omega t + 2\sum_{n=1}^{\infty} \frac{\sin n\omega t}{n \cosh\left[n\pi K'/K\right]}$$

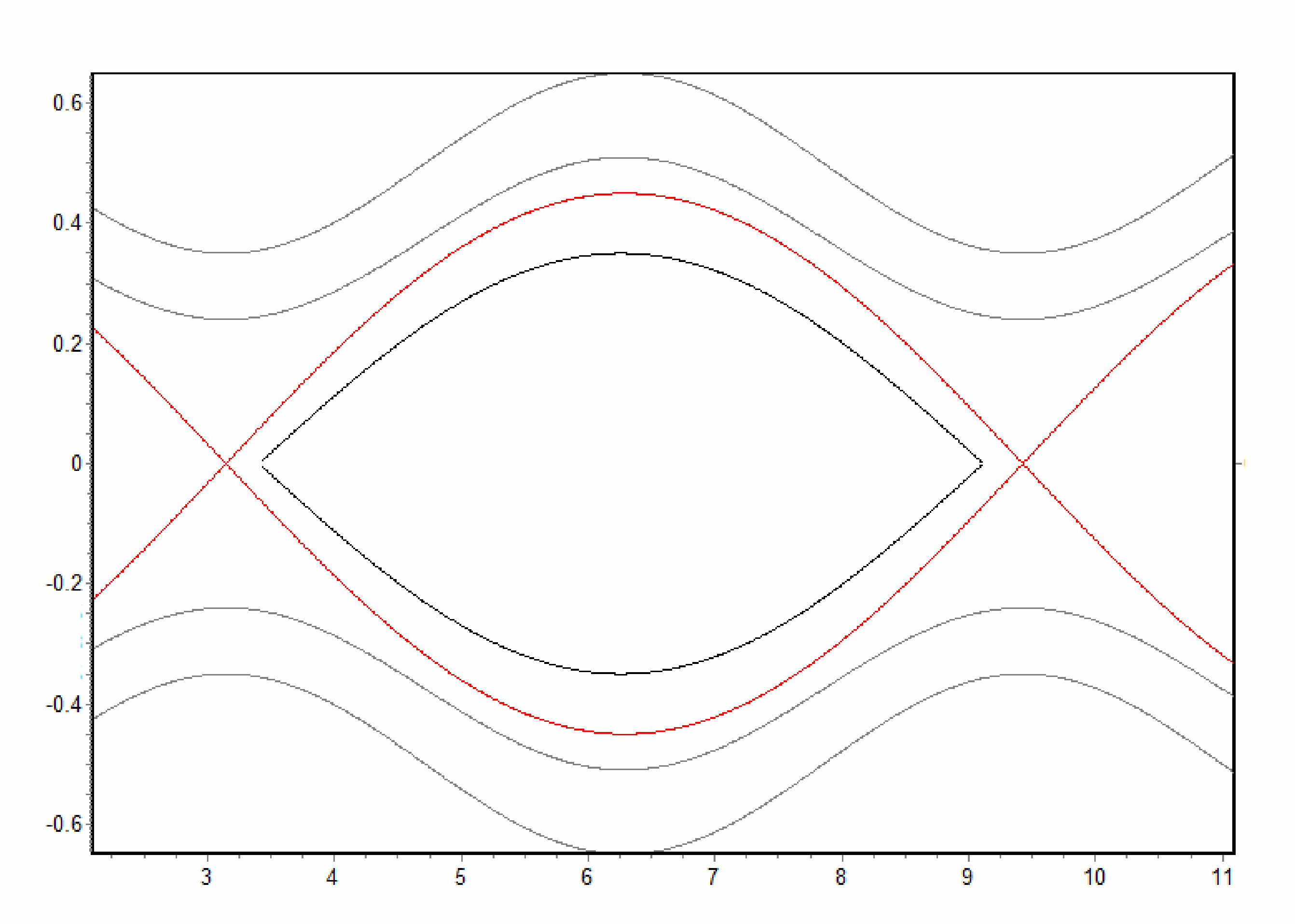
In the case of libration:

$$\varphi \equiv \sigma = 4\sum_{n=1}^{\infty} \frac{\sin(2n-1)\omega t}{(2n-1)\cosh\left[(n-1/2)\pi K'/K\right]}$$

where *K(k)* and *K'(k)* are the complete elliptic integrals of the first kind with modulus

$k_L = ((\mu A - H')/2\mu A)^{1/2}$ $\qquad$ $k_C = (2\mu A/(\mu A - H'))^{1/2}$

- The classical phase portrait is symmetric:



MD model

- However, in according with the classical theory, the probability of the resonance capture is depended on the direction of the resonance approach
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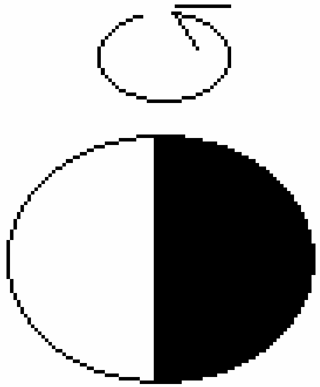
$$\delta = \frac{\alpha}{\beta\eta} = \alpha \left[\frac{4}{\varepsilon^2 \beta^{2-k}} \right]^{\frac{1}{4-k}} \tag{14}$$

The main result of the Murray and Dermott model is the following: when δ increased (diverged orbits), the capture in resonance is impossible. At the δ decreasing (converged orbits) capture is possible with finite probability (in dependence of the eccentricity value). At $e < e_{crit}$ the capture is inevitable.

$$e_{crit} = \sqrt{6} \left[\frac{3}{f_d} (1-j)^{4/3} j^{2/3} \frac{m_s}{m_p} \right]^{-1/3} \tag{15}$$

Yarkovsky effect



$$\frac{da}{dt} = \frac{\alpha\Phi}{n} f_P(\Theta) \cos\gamma ,$$

Numerical integration evidence of resonance asymmetry

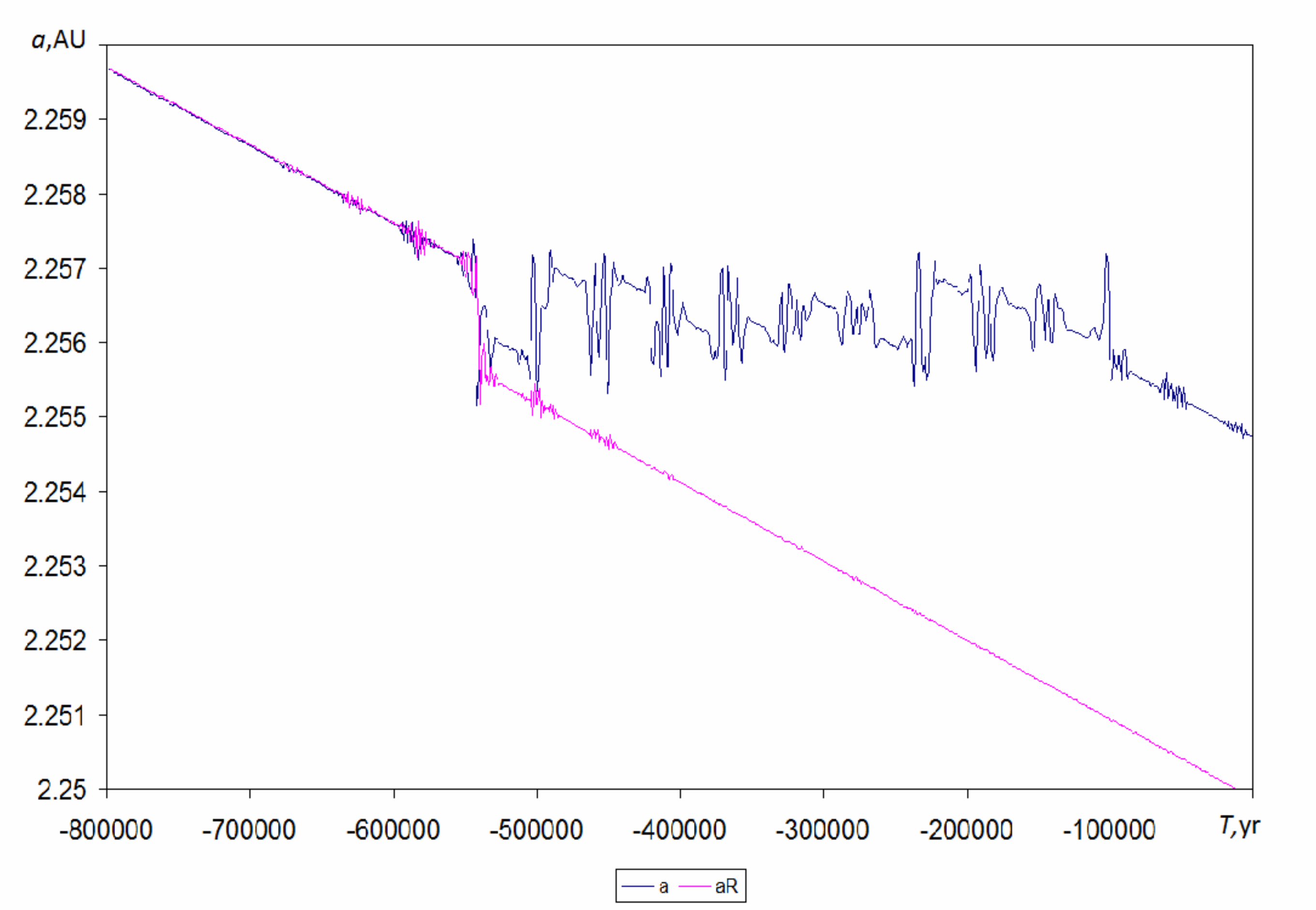


Fig.5. Forward and backward integration of orbit across 7:2J resonance

The reflection from resonance

- The MD model explain the process of transfer or capture in resonance. However in some cases the “reflection ” take place:

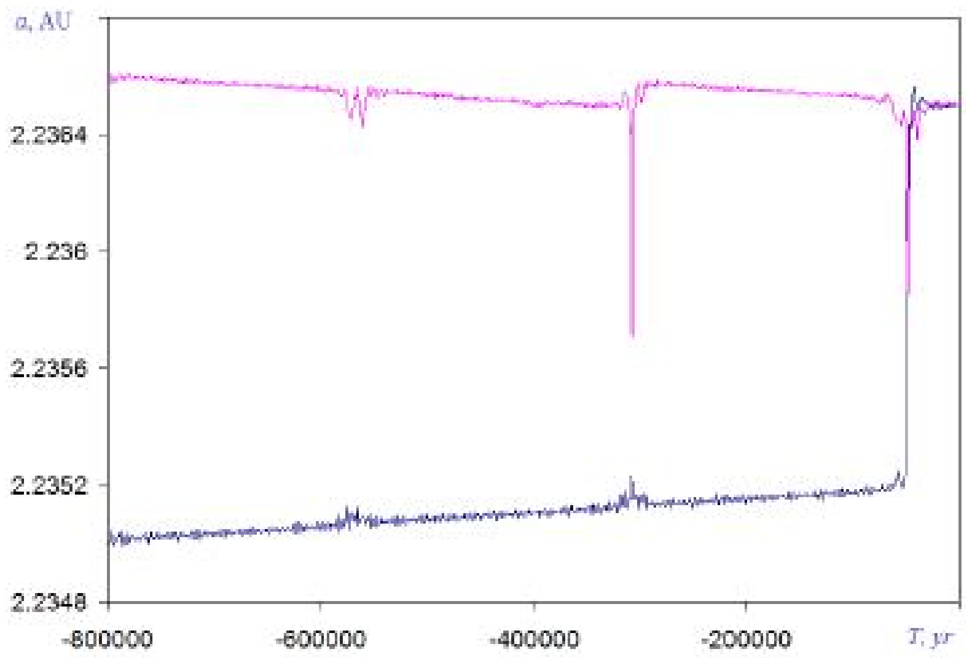
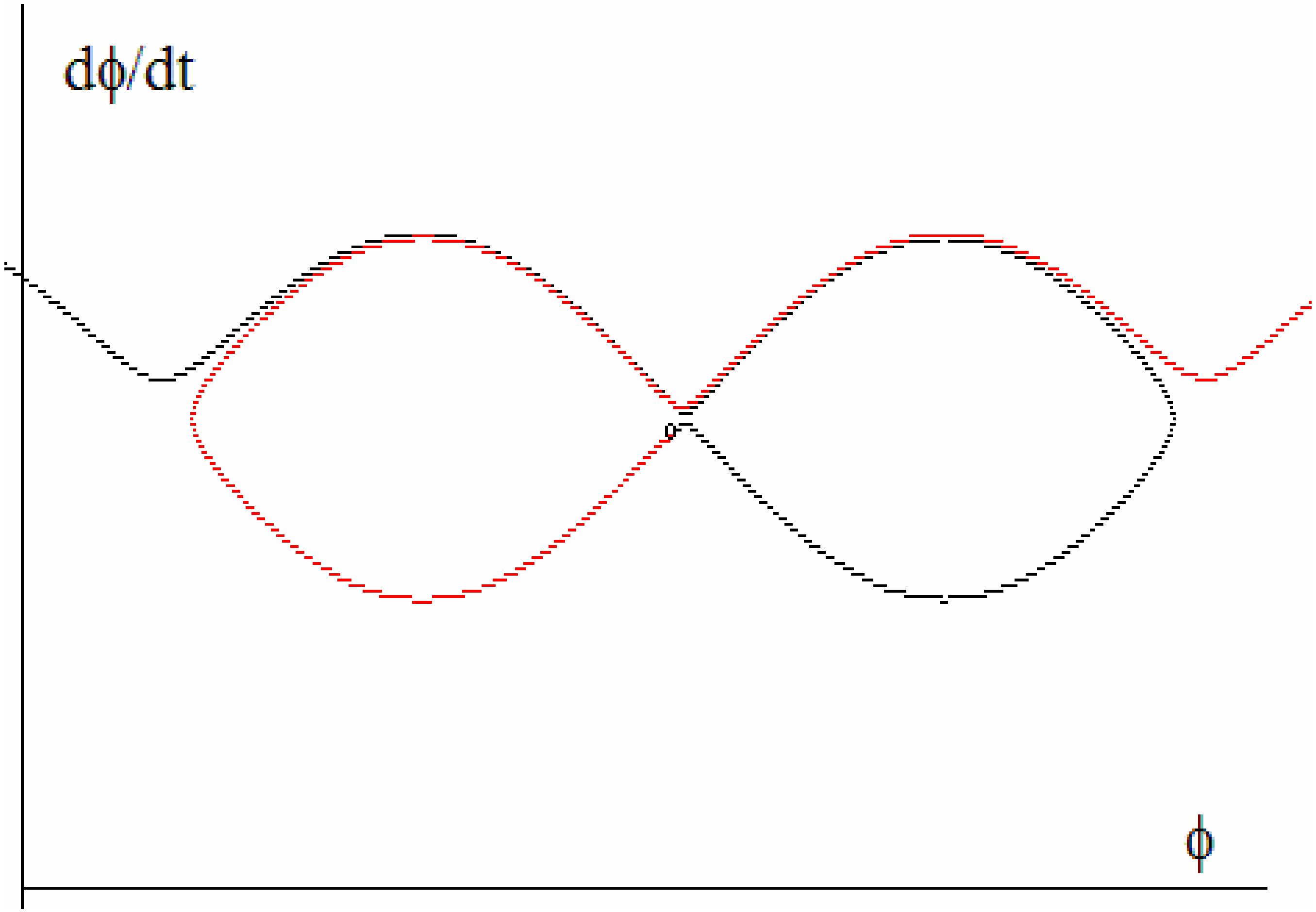


Fig. 1 The evolution of the semimajor axis of the asteroid 485010 2009 VS116 in the vicinity of the 9:16M resonance with different Yarkovsky drift.

Example of “reflection” in asymmetric model



Asymmetric model

The case of constant average angular velocity (circulation solution)

First, we can simply redraw the lines with the same absolute value of angular velocity in the classical phase portrait.

The exact solution of the pendulum equation is not suitable, because it contained the elliptic integral. By this reason, we consider asymptotic solution[2]. After the substitution $\dot{\lambda} = n - n_{res} = \delta_n$ for the circulation solution we have:

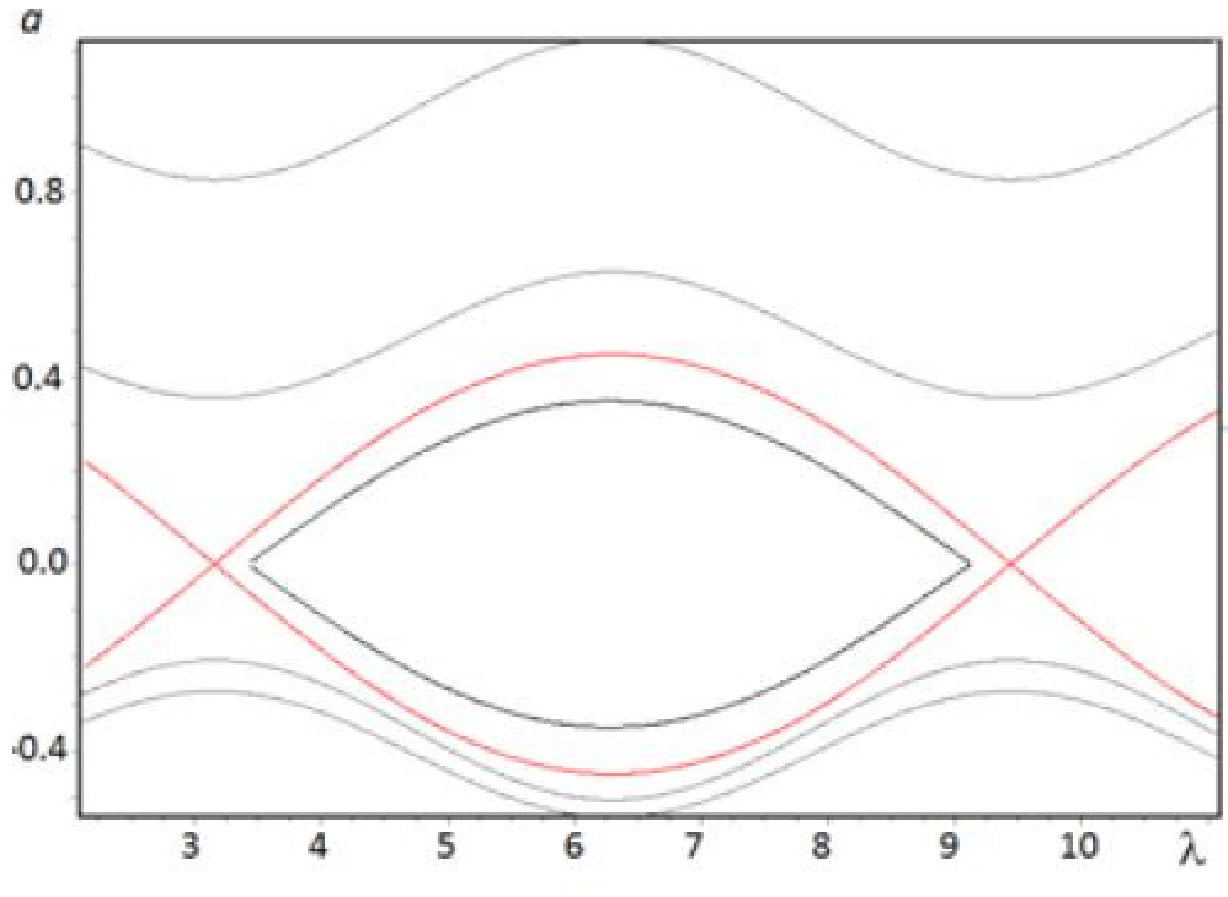
$$\dot{\lambda} = \Omega + \frac{\Omega}{\Omega^2}\cos\Omega t = \delta_n \pm \frac{1}{\delta_n}\cos\delta_n t \qquad (5)$$

Denote $a - a_{res} = \delta_a$ and using the expansion:

$$\delta_n = \pm 3/2\delta_a\frac{n_{res}}{a_{res}} + \frac{15}{8}\delta_a^2\frac{n_{res}}{a_{res}^2} + ... \qquad (6)$$

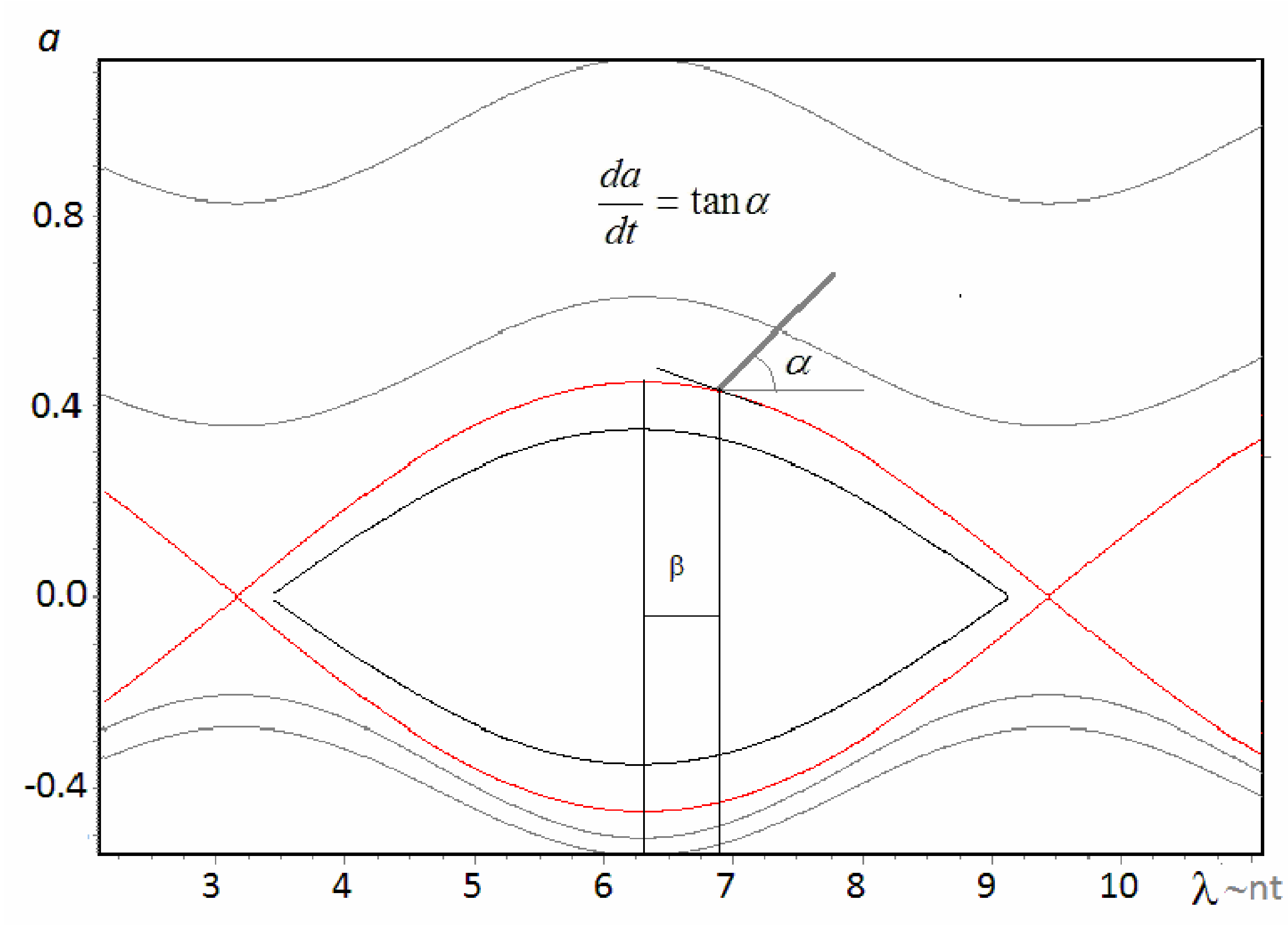
Finally we obtain:

$$\dot{\lambda} = a_{res}^{-3/2} \pm 3/2\delta_a\frac{n_{res}}{a_{res}} + \frac{15}{8}\delta_a^2\frac{n_{res}}{a_{res}^2} + ... \qquad (7)$$



The phase portrait of asymmetric pendulum in a, λ coordinates

Application for migration



Results and Possible future study

- It is easy to see that motion above and below of resonance is not the same due to the different gradient in a values
- It is interesting to study the process of the resonance approach in a, lambda coordinates as a function of alpha and the angle of contact separatrix (beta)

Another type of asymmetry

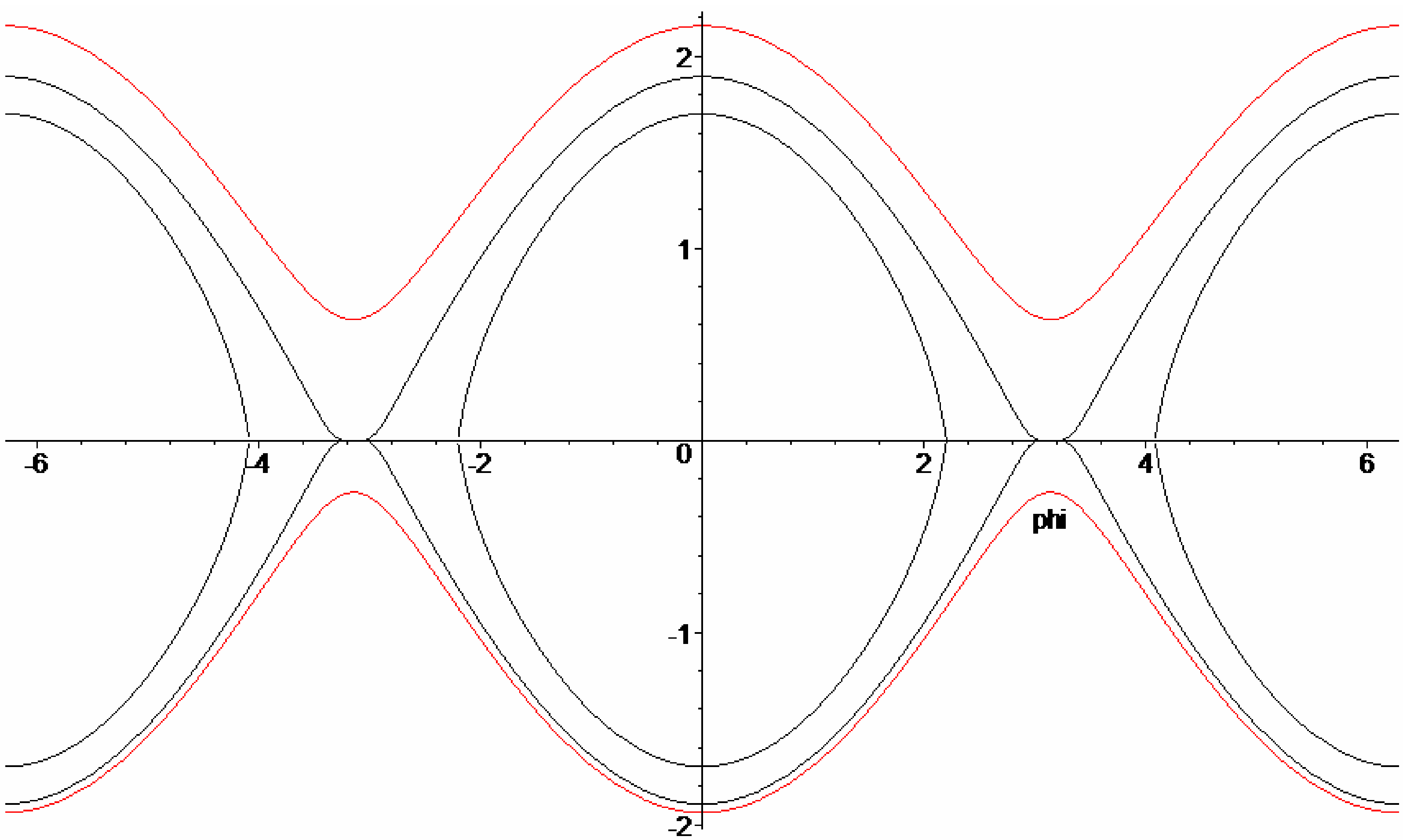
The case of fixed energy

According to the pendulum theory the relative angular velocity depends on the resonance energy (expression 8.48 in MD):

$$\frac{1}{2}\dot{\varphi}^2 + \omega_0^2(1-\cos\varphi) = E$$

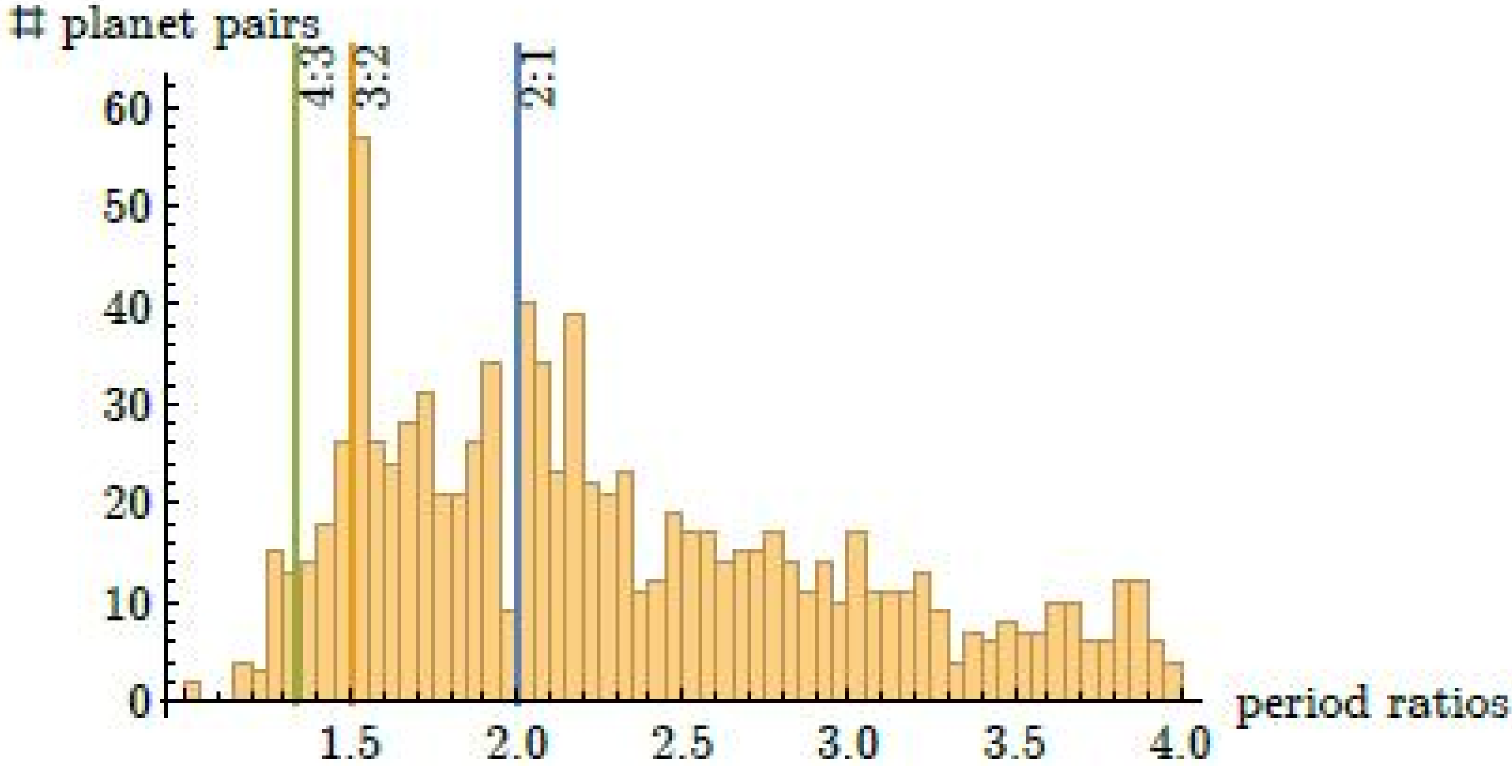
If we take into account the Keplerian energy, there is an asymmetry between the motion above (close to the central mass) and below (close to the perturbing planet) the resonance. In the case of the motion above the resonance, the average motion is slightly larger than in the exact resonance: $n = n_{res} + \delta_n$ Finally we have obtained:

$$\dot{\varphi}^2 + 2\omega_0^2(1-\cos\varphi) = 2E_0(1+\frac{2\dot{\varphi}}{3n_{res}})$$



The example of such “isoenergetic” asymmetry

Possible application



Conclusions

- The model of pendulum is widely used to study resonance in celestial mechanics. It is usually considered in (φ, $\dot{\varphi}$) coordinates where it is symmetric.
- However, the asymmetry occurs when we rebuild the phase portrait in (φ, a) coordinates, where φ is resonance argument, a is semi-major axis.
- This modification of the pendulum model has some advantages because the migration rate (and the rate of the resonance approach) is expressed in terms of a semi-major axis.

Some references:

- 1. Murray CD, Dermott SF. Solar system dynamics, Cambridge University Press, (1999). 606 p
- 2.Moiseev N. N., Asymptotic Methods in Nonlinear Mechanics [in Russian], Moscow, Nauka (1981).
- 3. Wisdom J. : A Perturbative Treatment of Motion near the 3/1 Commensurability., Icarus, 63, 272--289 (1985)
- 4.Pichierri G., Batygin K., Morbidelli A.: The role of dissipative evolution for three-planet, near-resonant extrasolar systems. A&A 625, A7 (2019)

4th International Conference on
INTEGRABLE SYSTEMS and NONLINEAR DYNAMICS
September 25-29, 2023, Yaroslavl, Russia

Venue:

The conference will be held in the city of Yaroslavl, 25-29 September 2023. The arrival day is September 24, 2023 and the departure day is September 29, 2023. The conference format will be mixed: in person as well as online for those participants who cannot travel to Russia.

Deadlines:

The registration is now open until June 15, 23:59 (GMT+3)

Conference site:

You can register and access all the conference les here:
https://lomonosov-msu.ru/eng/event/8178/

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Thank you!