Describing classicality of states of a finite-dimensional quantum system via Wigner function positivity

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Physical motivation

Classically, a particle in one dimension with its position q and momentum p is described by a phase space distribution $P_{Cl}(q,p)$. The average of a function of the position and momentum $A(q,p)$ can then be expressed as

$$\langle A\rangle_{Cl} = \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dp\, A(q,p)\, P_{Cl}(q,p)\,.$$

A **quantum mechanical** particle is described by a density matrix $\hat{\varrho}$ and the average of a function of the position and momentum operators $\hat{A}(\hat{q},\hat{p})$ is

$$\langle A\rangle_{QM} = \operatorname{tr}\left(\hat{A}\,\hat{\varrho}\right)\,.$$

A quantum mechanical average can be expressed using a quasiprobability distribution $P_{QM}(q,p)$ as

$$\langle A\rangle_{QM} = \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dp\, A(q,p)\, P_{QM}(q,p)\,.$$

Objective

Because of Heisenberg's uncertainty principle, the function $P_{QM}(q, p)$ has **negative values** for certain quantum states. Hence, it is not a true probability density and is referred to as a quasiprobability distribution.

Due to this negativity property, quasiprobability distributions may serve as a tool for understanding the interrelations between quantum and classical statistical descriptions.

Aim of the talk:

To consider the Wigner quasiprobability distribution $W(q, p)$ and, specifying the notion of "classical states" as the states whose Wigner function is non-negative everywhere in the phase space, to quantify a state classicality.

Wigner function

The Wigner quasiprobability distribution

$$W(\Omega_N) = \mathrm{tr}\,[\varrho\, \Delta(\Omega_N)]$$

is constructed from the **density matrix** (describing a quantum state)

$$\varrho \in \mathfrak{P}_N = \{X \in M_N(\mathbb{C}) \mid X = X^\dagger\,, \quad X \geq 0\,, \quad \mathrm{tr}\,(X) = 1\}$$

and the **Stratonovich-Weyl** self-dual **kernel**

$$\Delta(\Omega_N) \in \mathfrak{P}^*_N = \{X \in M_N(\mathbb{C}) \mid X = X^\dagger\,, \quad \mathrm{tr}\,(X) = 1\,, \quad \mathrm{tr}\left(X^2\right) = N\}\,,$$

defined over the symplectic manifold Ω_N .

Density matrix

A state of an N-**level quantum system** is given by the density matrix

$$\varrho = \frac{1}{N}\,\mathbb{I}_N + \sqrt{\frac{N-1}{2N}}\,(\boldsymbol{\alpha}, \boldsymbol{\lambda}) \,,$$

where $\boldsymbol{\alpha}$ is (N^2-1)-dimensional Bloch vector and $\boldsymbol{\lambda} = \{\lambda_1 \,, \cdots \,, \lambda_{N^2-1}\}$ is $\mathfrak{su}(N)$ algebra orthonormal Hermitian basis.

The singular value decomposition of the density matrix reads:

$$\varrho = U\,\mathrm{diag}\,(r_1, \dots, r_N)\; U^\dagger \,, \qquad U \in SU(N)\,,$$

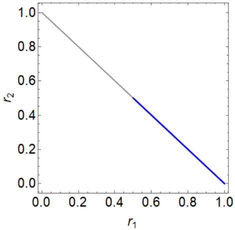
the spectrum $\{r_1, \dots, r_N\}$ of the density matrix forms Δ_{N-1}-simplex:

$$1 \geq r_1 \geq \cdots \geq r_N \geq 0 \,, \quad \sum_{i=1}^{N} r_i = 1 \,.$$

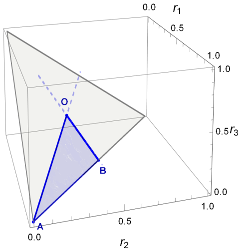
For $N = 2$ (qubit) Δ_1 : $\{1 \geq r_1 \geq r_2 \geq 0\,, \quad \sum_{i=1}^{2} r_i = 1\}$.

For $N = 3$ (qutrit) Δ_2 : $\{1 \geq r_1 \geq r_2 \geq r_3 \geq 0\,, \quad \sum_{i=1}^{3} r_i = 1\}$.

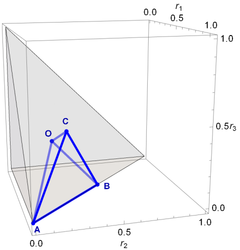
For $N = 4$ (quatrit) Δ_3 : $\{1 \geq r_1 \geq r_2 \geq r_3 \geq r_4 \geq 0\,, \quad \sum_{i=1}^{4} r_i = 1\}$.



$N = 2$



$N = 3$



$N = 4$

Stratonovich-Weyl kernel

The Stratonovich-Weyl kernel is the following:

$$\Delta(\Omega_N) = \frac{1}{N}\mathbb{I}_N + \sqrt{\frac{N^2-1}{2N}} \sum_{\lambda_s \in K} \mu_s \lambda_s \,,$$

$K \in \mathfrak{su}(N)$ is Cartan subalgebra, real coefficients $\sum_{s=2}^{N} \mu^2_{s^2-1} = 1\,.$

The SVD of the Stratonovich-Weyl kernel reads:

$$\Delta(\Omega_N) = V \operatorname{diag}(\pi_1, \dots, \pi_N)\; V^\dagger\,, \qquad V \in SU(N)\,.$$

Ordering of the spectrum $\{\pi_1, \dots, \pi_N\}$ of the SW kernel cuts out the moduli space of $\Delta(\Omega_N)$ in the form of a spherical polyhedron:

$$\pi_1 \geq \dots \geq \pi_N\,, \qquad \sum_{i=1}^{N} \pi_i = 1\,, \qquad \sum_{i=1}^{N} \pi_i^2 = N\,.$$

For $\kappa = \sqrt{\frac{N(N^2-1)}{2}}$ the SW kernel spectrum $\boldsymbol{\pi}$ may be presented as:

$$\pi_i = \frac{1}{N}\left(1 + \sqrt{2}\kappa \sum_{s=i+1}^{N} \frac{\mu_{s^2-1}}{\sqrt{s\,(s-1)}} - \kappa\sqrt{\frac{2\,(i-1)}{i}}\mu_{i^2-1}\right).$$

The conventional parameterization by $N-2$ spherical angles:

$$\mu_3 = \sin\psi_1 \cdots \sin\psi_{N-2}\,; \ldots\,;$$
$$\mu_{i^2-1} = \sin\psi_1 \cdots \sin\psi_{N-i}\cos\psi_{N-i+1}\,; \ldots\,;$$
$$\mu_{N^2-1} = \cos\psi_1\,, \qquad i = \overline{2, N}\,,$$

where for $\pi_1 \geq \cdots \geq \pi_N$ the constraints on μ_i are:

$$\mu_3 \geq 0\,, \qquad \mu_{(i+1)^2-1} \geq \sqrt{\frac{i-1}{i+1}}\,\mu_{i^2-1}\,, \quad i = \overline{2, N-1}\,.$$

Moduli space

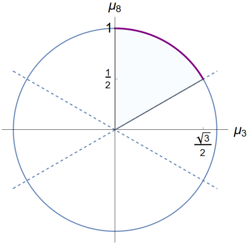
For $\boldsymbol{N=2}$: $\quad \pi_1 \geq \pi_2\,,\ \sum_{i=1}^{2} \pi_i = 1\,,\ \sum_{i=1}^{2} \pi_i = 2\,,$ so:

$$\pi_1 = (1+\sqrt{3})/2\,, \quad \pi_2 = (1-\sqrt{3})/2\,.$$

For $\boldsymbol{N=3}$: $\quad \pi_1 \geq \pi_2 \geq \pi_3\,,\ \sum_{i=1}^{3} \pi_i = 1\,,\ \sum_{i=1}^{3} \pi_i = 3,$ so:

$$\pi_2 = (1-\pi_1+\sqrt{5+2\pi_1-3\pi_1^2})/2\,, \quad 1 \leq \pi_1 \leq 5/3\,,$$

or, equivalently, for $\mu_3 = \sin\zeta\,,\ \mu_8 = \cos\zeta$: $\quad 0 \leq \zeta \leq \pi/3\,.$

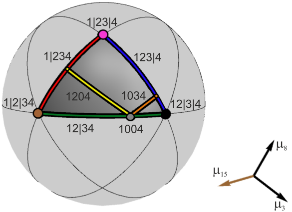


For $\boldsymbol{N = 4}$: $\pi_1 \geq \pi_2 \geq \pi_3 \geq \pi_4$, $\sum_{i=1}^{4} \pi_i = 1$, $\sum_{i=1}^{4} \pi_i = 4$, so for

$$\mu_3 = \sin\psi_1 \sin\psi_2\,, \quad \mu_8 = \sin\psi_1 \cos\psi_2\,, \quad \mu_{15} = \cos\psi_1\,,$$

where $\mu_3 \geq 0$, $\mu_8 \geq \frac{\mu_3}{\sqrt{3}}$, $\mu_{15} \geq \frac{\mu_8}{\sqrt{2}}$, the moduli space reads:

$$\left[\begin{array}{l} \begin{cases} \psi_2 \in \left(0, \frac{\pi}{3}\right]\,, \\ 0 < \psi_1 \leq arccot\left(\cos\psi_2/\sqrt{2}\right)\,; \end{cases} \\ \begin{cases} \psi_2 = 0\,, \\ 0 < \psi_1 \leq arccot\left(1/\sqrt{2}\right)\,; \end{cases} \\ \psi_1 = 0\,. \end{array}\right.$$



Quatrit moduli space as the Möbius spherical triangle $(2, 3, 3)$ on a unit sphere.

Wigner function positivity

A family of the Wigner functions:

$$W(\Omega_N) = \frac{1}{N}\left(1 + \frac{N^2-1}{\sqrt{N+1}}(\boldsymbol{n}, \boldsymbol{\alpha})\right) ,$$

vectors $\boldsymbol{n} = \mu_3 \boldsymbol{n}^{(3)} + \ldots + \mu_{N^2-1}\boldsymbol{n}^{(N^2-1)}$, $\boldsymbol{n}_\mu^{(s^2-1)} = \frac{1}{2}\,\mathrm{tr}\left(U\lambda_{s^2-1}U^\dagger\lambda_\mu\right)$.

For $\boldsymbol{r} \in \underline{\Delta}_N$, $\boldsymbol{\pi} \in \mathbf{spec}\left(\Delta(\Omega_N)\right)$, the lower bound of Wigner function

$$W_N^{(-)} = \sum_{i=1}^{N} \pi_i r_{N-i+1} = r_1\pi_N + \ldots + r_N\pi_1$$

determines the WF positivity region.

At that: $W_N^{(-)} \leq W(\Omega_N) \leq W_N^{(+)}$, $W_N^{(+)} = \sum_{i=1}^{N}\pi_i r_i$.

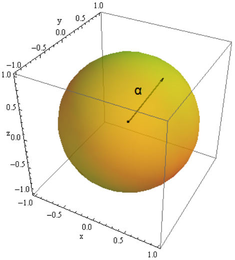
Qubit state

The state of a qubit is given by the density matrix

$$\varrho_2 = \frac{1}{2}\left(\mathbb{I}_2 + \boldsymbol{\alpha} \cdot \boldsymbol{\sigma}\right) = U \,\mathrm{diag}(r_1, r_2)\, U^\dagger = U\, \frac{1}{2}(\mathbb{I}_2 + r\sigma_3)\, U^\dagger \,,$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ is a Bloch vector, $r = |\boldsymbol{\alpha}|$, and $\boldsymbol{\sigma}$ is the basis of $\mathfrak{su}(2)$ algebra – the standart Pauli matrices.

Since $\varrho \geq 0$, the parameters space is restricted to the unit ball ($\boldsymbol{\alpha}^2 \leq 1$), and pure states describe the so-called Bloch sphere ($\boldsymbol{\alpha}^2 = 1$).



Qubit Wigner function lower bound: $W_2^{(-)} = r_1\pi_2 + r_2\pi_1\,.$

Qutrit state

A generic qutrit state is given by the density matrix

$$\varrho_3 = \frac{1}{3}(\mathbb{I}_3 + \sqrt{3}\sum_{\nu=1}^{8}\alpha_\nu\lambda_\nu) = U\,\mathrm{diag}(r_1, r_2, r_3)\,U^\dagger = U\,\frac{1}{3}(\mathbb{I}_3 + \sqrt{3}\sum_{i=3,8}\xi_i\lambda_i)\,U^\dagger\,,$$

where $\boldsymbol{\alpha}$ is an 8-dimensional Bloch vector, $\boldsymbol{\lambda} = \{\lambda_1\,, \cdots, \lambda_8\}$ is $\mathfrak{su}(3)$ algebra basis – the Gell-Mann matrices, and coefficients ξ_3, ξ_8 are invariants under the adjoint $SU(3)$ transformations of ϱ_3.

Qutrit Wigner function lower bound: $W_3^{(-)} = r_1\pi_3 + r_2\pi_2 + r_3\pi_1\,.$

Quatrit state

A generic quatrit state is given by the density matrix

$$\varrho_4 = \frac{1}{4}(\mathbb{I}_4 + \sqrt{6}\sum_{\nu=1}^{15}\alpha_\nu\lambda_\nu) = U\,\mathrm{diag}(r_1, r_2, r_3, r4)\,U^\dagger = U\,\frac{1}{4}(\mathbb{I}_3 + \sqrt{6}\sum_{i=3,8,15}\xi_i\lambda_i)\,U^\dagger\,,$$

where $\boldsymbol{\alpha}$ is a 15-dimensional Bloch vector, $\boldsymbol{\lambda} = \{\lambda_1\,,\cdots,\lambda_{15}\}$ is $\mathfrak{su}(4)$ algebra basis, and coefficients ξ_3, ξ_8, ξ_{15} are invariants under the adjoint $SU(4)$ transformations of ϱ_4 .

Quatrit WF lower bound: $W_4^{(-)} = r_1\pi_4 + r_2\pi_3 + r_3\pi_2 + r_4\pi_1\,.$

State space $\mathfrak{P}_N$

Unitary $U(N)$ automorphism of the Hilbert space of an N-level quantum system induces the adjoint $SU(N)$-action on state space $\mathfrak{P}_N$:

$$g \cdot \varrho = g\, \varrho\, g^\dagger\,, \qquad g \in SU(N)\,,$$

which sets equivalence relations between elements of $\mathfrak{P}_N$ and gives rise to its decomposition over the strata:

$$\mathfrak{P}_{[H_\alpha]} := \{x \in \mathfrak{P}_N |\ H_x \text{ is conjugate to } H_\alpha\}\,, \quad \mathfrak{P}_N = \bigcup_{\text{orbit types}} \mathfrak{P}_{[H_\alpha]}\,.$$

A subgroup $H_x \subset SU(N)$ is the isotropy group of a point $x \in \mathfrak{P}_N$,

$$H_x = \{g \in SU(N) \mid g \cdot x = x\}\,,$$

and points $x\,, y \in \mathfrak{P}_N$ are said to be of the same type if their stabilizers H_x and H_y are conjugate subgroups of $SU(N)$ group.

The "classical states" form the subset $\mathfrak{P}_N^{(+)} \subset \mathfrak{P}_N$ of states whose Wigner function is non-negative everywhere over the phase space:

$$\mathfrak{P}_N^{(+)} = \{ \varrho \in \mathfrak{P}_N \mid W_\varrho(z) \geq 0 \,, \quad \forall z \in \Omega_N \} \,.$$

The "classical states on a fixed stratum" $\mathfrak{P}_{H_\alpha}$ are defined as:

$$\mathfrak{P}_{H_\alpha}^{(+)} = \mathfrak{P}_N^{(+)} \cap \mathfrak{P}_{H_\alpha} \,.$$

The unitary orbit space $\mathcal{O}[\mathfrak{P}_N]$ is the quotient space under the equivalence relation imposed by the adjoint $SU(N)$-action on the state space $\mathfrak{P}_N$ with quotient mapping $\pi : \mathfrak{P}_N \longrightarrow \mathcal{O}[\mathfrak{P}_N] = \mathfrak{P}_N / SU(N)$.

The subset $\mathcal{O}[\mathfrak{P}_N^{(+)}] = \pi[\mathfrak{P}_N^{(+)}] = \{\pi(x) \mid x \in \mathfrak{P}_N^{(+)}\}$ represents the image of $\mathfrak{P}_N^{(+)}$ under the quotient mapping π.

Nonclassicality characteristics of states

Nonclassicality measures based on the violation of the Wigner function semi-positivity can be divided into different types:

1. (Nonclassicality distance) based on a **distance** of a state from the "classical states":

$$d_\varrho = \inf_{x \in \mathfrak{P}_N^{(+)}} D(\varrho, x) = \sqrt{\inf_{x_{diag} \in \mathcal{O}[\mathfrak{P}_N^{(+)}]} \sum_{i=1}^{N} (r_i - x_i)^2}\,,$$

where states with positive Wigner functions are taken as the reference "classical states", $\mathfrak{P}_N^{(+)}$.

2. (Kenfack-Życzkowski indicator) based on the **volume** of a phase space region where the Wigner function is negative:

$$\delta_N = \int_{\Omega_N} \mathrm{d}\Omega_N \left| W(\Omega_N) \right| - 1\,.$$

Qubit nonclassicality distance and KZ-indicator

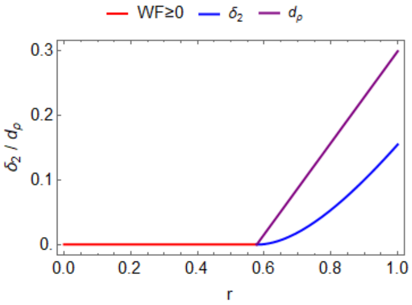
Qubit Wigner function: $W(\Omega_2) = \frac{1}{2}\left(1 + \sqrt{3}\,(\boldsymbol{n}, \boldsymbol{\alpha})\right).$

Qubit nonclassicality distance for Hilbert-Schmidt metric:

$$d_\varrho = \theta[\mathrm{r} - \tfrac{1}{\sqrt{3}}]\left(\frac{\mathrm{r}}{\sqrt{2}} - \frac{1}{\sqrt{6}}\right).$$

Qubit KZ-indicator:

$$\delta_2 = \theta[\mathrm{r} - \tfrac{1}{\sqrt{3}}]\left(\frac{3\mathrm{r}^2+1}{2\sqrt{3}\mathrm{r}} - 1\right).$$



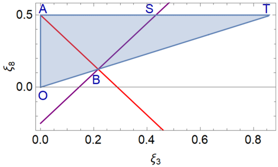
Qutrit nonclassicality distance

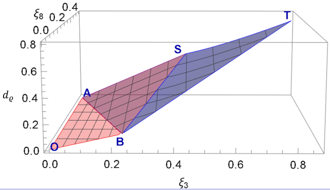
Qutrit Wigner function: $W(\Omega_3) = \frac{1}{3}\left(1 + 4\left(\boldsymbol{n}, \boldsymbol{\alpha}\right)\right).$

Qutrit nonclassicality distance for Hilbert-Schmidt metric:

$$d_\varrho = \begin{cases} 0\,, & \text{if} \quad \xi_3, \xi_8 \in \triangle OAB\,, \\ \frac{1}{4}\left|2\xi_3 \csc\left(\zeta + \frac{\pi}{6}\right) + 2\xi_8 \sec\left(\zeta + \frac{\pi}{6}\right) - \sec\left(2\zeta - \frac{\pi}{6}\right)\right|\,, & \text{if} \quad \xi_3, \xi_8 \in \triangle ABS\,, \\ \sqrt{\left(\xi_3 - \frac{\sqrt{3}}{8}\sec(\zeta)\right)^2 + \left(\xi_8 - \frac{\sec(\zeta)}{8}\right)^2}\,, & \text{if} \quad \xi_3, \xi_8 \in \triangle BST\,. \end{cases}$$

Qutrit $\underline{\Delta}_2$-simplex with WF positivity boundary and nonclassicality distance ($\zeta = 0$):





Qutrit KZ-indicator

Qutrit KZ-indicator for moduli parameter $\zeta = 0$:

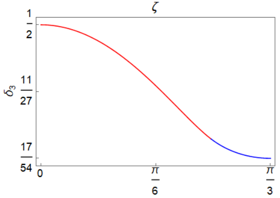
$$\delta_{(1|23)}(\boldsymbol{\xi}_{\mathrm{d}}\,|\,0) = \begin{cases} 0\,, & \text{if} \quad \xi_3, \xi_8 \in \triangle OAP\,, \\ \dfrac{1}{36}\,\dfrac{(2(\sqrt{3}\xi_3 + \xi_8) - 1)^3}{\xi_3(\xi_3 + \sqrt{3}\xi_8)}\,, & \text{if} \quad \xi_3, \xi_8 \in \triangle APC\,. \end{cases}$$

Qutrit KZ-indicator for moduli parameter $\zeta = \pi/3$:

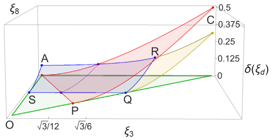
$$\delta_{(12|3)}(\boldsymbol{\xi}_{\mathrm{d}}\,|\,\frac{\pi}{3}) = \begin{cases} 0\,, & \text{if} \quad \xi_3, \xi_8 \in \triangle OSQ\,, \\ \dfrac{1}{18}\,\dfrac{(1 - 4\xi_8)^3}{(\xi_3^2 - 3\xi_8^2)}\,, & \text{if} \quad \xi_3, \xi_8 \in \square ARQS\,, \\ \dfrac{1}{36}\,\dfrac{\left(2(\sqrt{3}\,\xi_3 + \xi_8) + 1\right)^3}{\xi_3(\xi_3 + \sqrt{3}\,\xi_8)} - 2\,, & \text{if} \quad \xi_3, \xi_8 \in \triangle CQR\,. \end{cases}$$

Qutrit KZ-indicator for pure states:

$$\delta_3 = \begin{cases} \dfrac{(-1+4\cos(\zeta))^3}{18(1+2\cos(2\zeta))}\,, & \text{if } 0 \leq \zeta \leq 2\arctan\left(\dfrac{\sqrt{3}}{2+\sqrt{5}}\right)\,, \\[2ex] \dfrac{\left(4\sin\left(\zeta+\frac{\pi}{6}\right)+1\right)^3}{18\left(1-2\cos\left(2\left(\zeta+\frac{\pi}{6}\right)\right)\right)}-2\,, & \text{if } 2\arctan\left(\dfrac{\sqrt{3}}{2+\sqrt{5}}\right) \leq \zeta \leq \dfrac{\pi}{3}\,. \end{cases}$$



The KZ-indicator as function of moduli parameter ζ for qutrit pure states.



Qutrit KZ-indicators $\delta_3^{(0)}$ (red surface) and $\delta_3^{(\frac{\pi}{3})}$ (blue and yellow surfaces) as functions of two invariants ξ_3 and ξ_8 .

The global indicator of classicality

3. (Global indicator of classicality) as the **relative volume** of a subspace $\mathfrak{P}_N^{(+)} \subset \mathfrak{P}_N$ of the state space $\mathfrak{P}_N$, consisting of states whose Wigner functions are **positive**:

$$\mathcal{Q}_N = \frac{\mathrm{Volume(Classical\ States)}}{\mathrm{Volume(All\ States)}},$$

where the Riemannian volume is calculated with respect to the measure dictated by the probability distribution function of an ensemble.

For classical states on the fixed stratum $\mathfrak{P}_{H_\alpha}$ the $Q-$indicator of classicality of the stratum is defined as:

$$\mathcal{Q}_N[H_\alpha] = \frac{\mathrm{Volume(Classical\ States\ on\ }\mathfrak{P}_{[\mathrm{H}_\alpha]})}{\mathrm{Volume(All\ States\ on\ }\mathfrak{P}_{[\mathrm{H}_\alpha]})}.$$

The Hilbert-Schmidt ensemble of qudits

If the full rank density matrix has a spectrum of the form

$$\boldsymbol{r}^{\downarrow}(\varrho) = \{r_1 \overbrace{(1,\dots,1)}^{k_1};\ r_2 \overbrace{(1,\dots,1)}^{k_2};\ \dots\ ;\ r_s \overbrace{(1,\dots,1)}^{k_s}\}$$

with N distinct non-zero eigenvalues $(k_1 = k_2 = \cdots = k_N = 1)$, then the metric corresponding to the distance between two infinitesimally close matrices $\varrho - \mathrm{d}\varrho$ and $\varrho + \mathrm{d}\varrho$ defines the standard Hilbert-Schmidt ensemble of random full rank $N-$qudits.

The joint probability distribution of eigenvalues then reads:

$$P^{\mathrm{HS}}(r_1,\dots,r_N) \propto \delta(1 - \sum_{j=1}^{N} r_j) \prod_{j<k}^{N} (r_j - r_k)^2 \,.$$

Degenerate Hilbert-Schmidt qudits

If the full rank density matrix spectrum has an arbitrary algebraic degeneracy $\boldsymbol{k} = (k_1, k_2, \dots, k_s)$, then the joint probability distribution of eigenvalues is reduced to the following expression:

$$P^{\mathrm{HS}}_{k_1,\dots,k_s}(r_1, \dots, r_s) \propto \delta(1 - \sum_{i=1}^{s} k_i r_i) \prod_{i<j}^{1\dots s} (r_i - r_j)^{2k_i k_j} \,.$$

Wherein the angles in the singular value decomposition are distributed according to the Haar measure on the coset

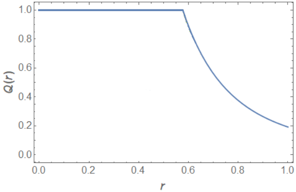
$$U(N)/U(k_1) \times \cdots \times U(k_s) \,.$$

Qubit global indicator of classicality

Wigner function $W(\Omega_2) \geq 0$ inside the Bloch ball of radius $\frac{1}{\sqrt{3}}$.

For the Hilbert-Schmidt ensemble of qubits the PDF $P^{\mathrm{HS}}(r) \propto r^2$, the global $\mathcal{Q}$-indicator of classicality:

$$\mathcal{Q}_2 = \frac{1}{3\sqrt{3}} \approx 0.19245\,.$$



The probability $Q(r) = \frac{\mathrm{vol}(\mathbb{B}(r)) \cap \mathcal{O}[\mathfrak{P}_2^+]}{\mathrm{vol}(\mathbb{B}(r))}$ to find a qubit state with WF$\geq$ 0 within the Bloch ball of radius r .

Qutrit global indicator of classicality

Qutrit orbit space and its subspace of WF positivity are respectively

$$\mathcal{O}[\mathfrak{P}_3] : \ \left\{ \boldsymbol{r} \in \mathbb{R}^2 \ \middle| \ \sum_{i=1}^{3} r_i = 1, \quad 1 \geq r_1 \geq r_2 \geq r_3 \geq 0 \right\},$$

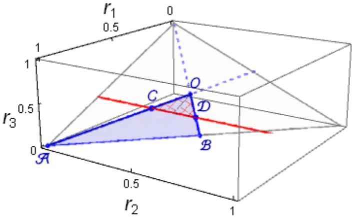
$$\mathcal{O}[\mathfrak{P}_3^{(+)}] : \left\{ \zeta \in [0, \pi/3] \ \middle| \ r_3 \geq \frac{r_1(4\cos\zeta - 1) - r_2\left(4\cos\left(\zeta + \frac{\pi}{3}\right) + 1\right)}{1 + 4\cos\left(\zeta - \frac{\pi}{3}\right)} \right\}.$$

Regular stratum $\mathcal{Q}$-indicator:

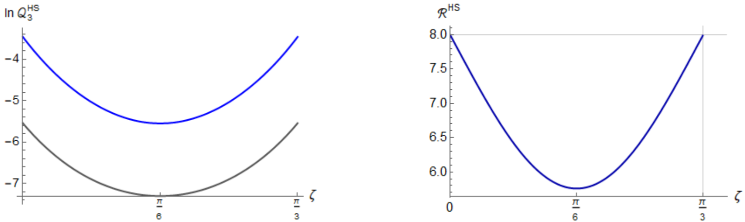
$$\mathcal{Q}_3 = \frac{20\cos^2(\zeta - \pi/6) + 1}{128\,(4\cos^2(\zeta - \pi/6) - 1)^5}.$$

Degenerate stratum $\mathcal{Q}$-indicator:

$$\mathcal{Q}_3^{H_{S(U(2)\times U(1))}} = \frac{\csc^5\left(\zeta + \frac{\pi}{6}\right) + \sec^5(\zeta)}{1056}.$$



The ratio $\mathrm{R}(\zeta) = \dfrac{\mathcal{Q}_3^{H_{S(U(2)\times U(1))}}(\zeta)}{\mathcal{Q}_3(\zeta)}$ may serve as a certain measure of relation between the symmetry of a state and its classicality.



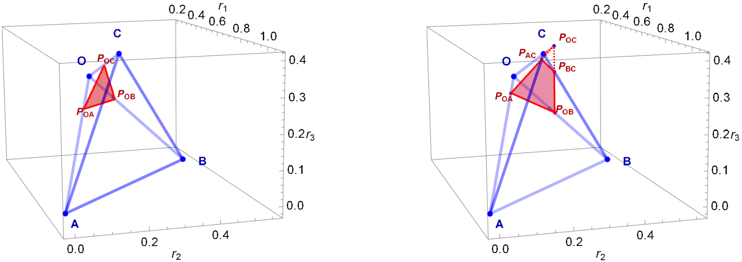
(a) $\mathcal{Q}_3-$indicators of a Hilbert-Schmidt qutrit as functions of ζ for the regular (gray curve) and degenerate (blue curve) strata. The absolute minimum of both indicators is attained at $\zeta = \pi/6$. (b) The ratio of degenerate to regular $\mathcal{Q}_3-$indicators.

Notation: the degenerate stratum $\mathfrak{P}_{[S(U(2)\times U(1))]}$ has two pieces, $F_{1|23}$ and $F_{12|3}$, associated with density matrices with degenerate eigenvalues $r_1 = r_2 \neq r_3$ and $r_1 \neq r_2 = r_3$ of types $\boldsymbol{k} = (1,2)$ and $\boldsymbol{k} = (2,1)$, respectively.

Quatrit global indicator of classicality

In order to find the subset of classical states, one has to analyse the intersections of a quatrit simplex – the tetrahedron $OCAB$ with the hyperplane $\pi_1 = (\pi_1 - \pi_4)r_1 + (\pi_1 - \pi_3)r_2 + (\pi_1 - \pi_2)r_3$.

There are only two types of admissible cross-sections:



(a) **triangles**, if the intersection points belong to edges of the tetrahedron emanating from vertex of maximally mixed states, $\pi_1 \geq 1$; (b) **quadrilaterals**, if an intersection point lies outside the edge of the tetrahedron, $\frac{1}{4} \leq \pi_1 < 1$.

Lasserre method for calculations

J.Lasserre (2021): integrating a polynomial of degree q on an arbitrary simplex (with respect to Lebesgue measure) reduces to evaluating q homogeneous polynomials of degree $j = 1, 2, \dots q$ each at a unique point $\boldsymbol{s}_j$ of the simplex.

Let $p(\boldsymbol{x}) = \sum_{j=0}^{q} p_j(\boldsymbol{x})$ be real polynomial of degree q ; $\boldsymbol{x} = (x_1, \dots, x_n)$ and $p_j(\boldsymbol{x}) = \sum_{|\alpha|=j} p_\alpha \boldsymbol{x}^\alpha$ is homogeneous polynomial of degree j .

Then the integration over the canonical $n-$dimensional simplex K_n :

$$\int_{K_n} p(\boldsymbol{y})\mathrm{d}\boldsymbol{y} = \mathrm{vol}(K)\left(\hat{p}_0 + \sum_{j=1}^{q} \hat{p}_j(\boldsymbol{s}_j)\right) ,$$

where $\boldsymbol{s}_j = \frac{(1,\dots,1)}{\sqrt[j]{(n+1)\dots(n+j)}}$ and $\hat{p}(\boldsymbol{x}) = \sum_{\alpha\in\mathbb{N}^n} p_\alpha \alpha_1! \dots \alpha_n! \, \boldsymbol{x}^\alpha$, with $\alpha = (\alpha_1, \dots, \alpha_n)$, is the associated **“Bombieri” polynomial**.

For a quatrit $(N = 4)$: $\quad \boldsymbol{s}_{12} = \left(\frac{6}{15!}\right)^{1/12} (1,1,1)\,,$ and the regular stratum $\mathcal{Q}$-indicator $\quad \mathcal{Q}_4 \propto \mathrm{vol}_{OP_{OC}P_{OA}P_{OB}} - \theta[1-\pi_1]\,\mathrm{vol}_{CP_{OC}P_{AC}P_{BC}}\,.$

The degenerate stratum $\mathcal{Q}$-indicator:

$$\mathcal{Q}_4^{H_{S(U(3)\times U(1))}} \propto \begin{cases} \frac{1}{1+3^7}\left(1+\frac{3^7}{(1-4\pi_4)^7}\right), & \pi_4 \leq 0\,,\ \frac{1}{4} < \pi_1 \leq 1\,, \\ \frac{3^7}{1+3^7}\left(\frac{1}{(4\pi_1-1)^7}+\frac{1}{(1-4\pi_4)^7}\right), & \pi_4 \leq 0\,,\ \pi_1 > 1\,. \end{cases}$$

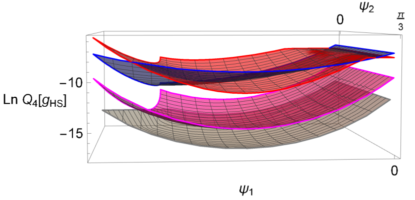
$$\mathcal{Q}_4^{1|23|4} \propto \begin{cases} 0\,, & \pi_1+\pi_2 \leq 1\,, \\ \frac{1}{(2\pi_1+2\pi_2-1)^9}\,, & \pi_1+\pi_2 > 1\,. \end{cases}$$

$$\mathcal{Q}_4^{H_{S(U(2)\times U(1)^2)}} \propto \begin{cases} f_1(\pi_1,\pi_2,\pi_4)\,, & \pi_1 > 1\,, \\ f_2(\pi_1,\pi_2)\,, & \frac{1}{4} < \pi_1 < 1\,, \pi_2 > \frac{1}{4}\,, \\ f_3(\pi_1,\pi_3,\pi_4)\,, & \pi_1 < 1\,, \pi_4 < 0\,. \end{cases}$$

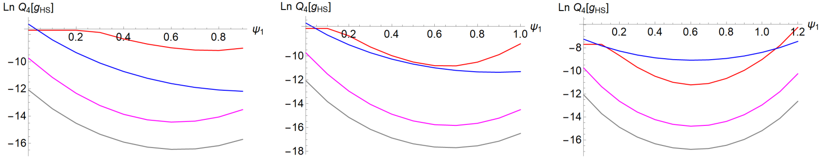
Notations: 1. $S(U(3)\times U(1))$: $1|2|34$ and $12|3|4$ of types $\boldsymbol{k} = (3,1)$ and $\boldsymbol{k} = (1,3)$, 2. $1|23|4$ of type $\boldsymbol{k} = (2,2)$, 3. $S(U(2)\times U(1)^2)$: $1|234$, $12|34$ and $123|4$ of types $\boldsymbol{k} = (2,1,1)$, $\boldsymbol{k} = (1,2,1)$ and $\boldsymbol{k} = (1,1,2)$.

$\mathcal{Q}$-indicators of a qubit-qubit system

Qubit-qubit global indicators for stratum of orbits with corresponding isotropy groups: $\mathcal{Q}_4$, $\mathcal{Q}_4^{H_{S(U(3)\times U(1))}}$, $\mathcal{Q}_4^{1|23|4}$, $\mathcal{Q}_4^{H_{S(U(2)\times U(1)^2)}}$



Slices of global indicators of classicality for different types of orbits of a qubit-qubit system for Hilbert-Schmidt metric:



$\psi_2 = 0$ $\psi_2 = \pi/6$ $\psi_2 = \pi/3$

Results

Three measures of classicality constructed out of the quasiprobability distributions were calculated for low-dimensional quantum systems:

- Nonclassicality distance d_ϱ ,
- Kenfack-Życzkowski indicator δ_N ,
- Global indicator $\mathcal{Q}$ both for regular and degenerate stratum.

It is intriguing that the global indicator $\mathcal{Q}$ in Hilbert-Schmidt metric as an integral over a simplex may be evaluated as a sum of certain permanents at the vertices of $\mathcal{O}[\mathfrak{P}_N^+]$.

CONJECTURE: more symmetry – more classicality!

Let us arrange the isotropy groups H_α in ascending order, starting from the maximal torus T_N up to the whole group $SU(N)$ [a],

$$T_N = H_{\min} < H_1 < \cdots < H_{\max} = SU(N)\,.$$

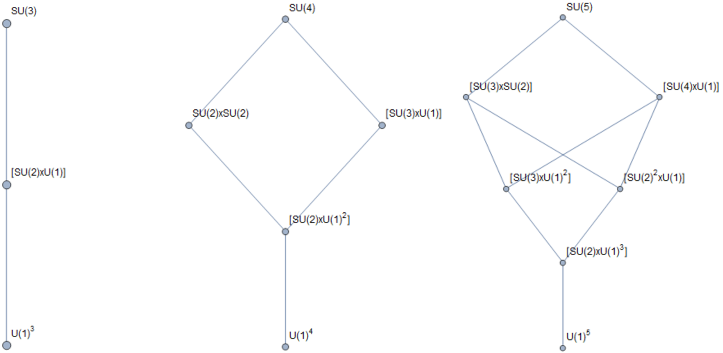
Then

$$\mathcal{Q}_N[T_N] < \mathcal{Q}_N[H_1] < \cdots < \mathcal{Q}_N[SU(N)] = 1\,.$$

[a] If H and K are isotropy subgroups of G, we define a partial ordering on equivalence classes by writing $(H) < (K)$ if H is G-conjugate to a subgroup of K. This defines a partial ordering on the set of isotropy types.

Hasse diagrams

Hasse diagram as a graphical representation of the relation of elements of a partially ordered set with an implied upward orientation:



$N = 3$ $\qquad$ $N = 4$ $\qquad$ $N = 5$

Thank you!