Bounded elementary generation of Chevalley groups and Steinberg groups

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Abstract. This is a sequel of our talk at the PCA-2022, see [17] Here we state a definitive result which almost completely closes the problem of bounded elementary generation for Chevalley groups over arbitrary Dedekind rings of arithmetic type with uniform bounds. Namely, for every reduced irreducible root system Φ of rank ≥ 2 there exists a universal bound $L = L(\Phi)$ such that the simply connected Chevalley groups $G(\Phi, R)$ have elementary width $\leq L$ for all Dedekind rings of arithmetic type R. We also state two results concerning bounded elementary generation of the corresponding Steinberg groups $\mathrm{St}(\Phi, R)$.

Introduction

In the present talk, we consider Chevalley groups $G = G(\Phi, R)$, their elementary subgroups $E(\Phi, R)$, and the corresponding Steinberg groups $\mathrm{St}(\Phi, R)$. over various classes of rings, mostly over Dedekind rings of arithmetic type (we refer to [40] for notation and further references pertaining to Chevalley groups, and to [2] for the number theory background).

Primarily, we are interested in the classical problem of estimating the width of $E(\Phi, R)$ with respect to the elementary generators $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in R$. We say that a group G is **boundedly elementarily generated** if $E(\Phi, R)$ has finite width $w_E(G)$ with respect to elementary generators.

This problem has attracted considerable attention over the last 40 years or so. Below, we reproduce the survey page from [17].

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• Bounded elementary generation always holds with obvious *small* bounds for 0-dimensional rings. This follows from the existence of such short factorisations as Bruhat decomposition, Gauß decomposition, unitriangular factorisation of length 4, and the like. On the other hand, bounded generation usually fails for rings of dimension ≥ 2. But for 1-dimensional rings it is problematic.

• Existence of arbitrary long division chains in Euclidean algorithm implies that $\mathrm{SL}(2,\mathbb{Z})$ and $\mathrm{SL}(2,\mathbb{F}_q[t])$ are not boundedly elementary generated [7]. But this could be attributed to the exceptional behaviours of rank 1 groups.

• What came as a shock, was when Wilberd van der Kallen [15] established that bounded elementary generation — and thus also finite commutator width — fail even for $\mathrm{SL}(3,\mathbb{C}[x])$, a group of Lie rank 2 over a Euclidean ring! Compare also [9], for a slightly simplified proof.

An emblematic example of 1-dimensional rings are Dedekind rings of arithmetic type $R = \mathcal{O}_S$, for which bounded elementary generation of $G(\Phi, R)$ is intrinsically related to the positive solution of the congruence subgroup problem in that group.

For the **number case** the situation is well understood, even for rank 1 groups. Without attempting to give a detailed survey, let us mention some high points of this development. Apart from the rings $R = \mathcal{O}_S$, $|S| = 1$, with finite multiplicative group, such finiteness results are even available for $\mathrm{SL}(2, R)$.

• For all Chevalley groups of rank ≥ 2, after the initial breakthrough by Douglas Carter and Gordon Keller, [4, 5], later explained and expanded by Oleg Tavgen [37], and many others, we now know bounded elementary generation with excellent bounds depending on the type of Φ and the class number of R alone.

This leaves us with the analysis of the group $\mathrm{SL}(2, R)$, for a Dedekind ring $R = \mathcal{O}_S$, with infinite multiplicative group.

• At about the same time, jointly with Paige, Carter and Keller gave a model theoretic proof [unpublished], [6], somewhat refashioned by Dave Morris [27]. But as all model theoretic proofs, this proof gives no bounds whatsoever.

• On the other hand, another important advance was made by Cooke and Weinberger [8], who got excellent bounds, modulo the Generalised Riemann Hypothesis. The explicit unconditional bounds obtained thereafter seemed to be grossly exaggerated [23].

• Some 10 years ago Maxim Vsemirnov and Sury [43] considered the key example of $\mathrm{SL}\left(2,\mathbb{Z}\left[\frac{1}{p}\right]\right)$, obtaining the bound $w_E(\mathrm{SL}(2,R)) = 5$ *unconditionally*.

• This was a key inroad to the first complete unconditional solution of the general case with a good bound, in the work of Alexander Morgan, Andrei Rapinchuk and Sury [25]. The bound they gave is ≤ 9, but for the case when S contains at least one real or non-Archimedean valuation was almost immediately improved [with the same ideas] to ≤ 8 by Jordan and Zaytman [13].

However, the **function case** turned out to be much more recalcitrant, and was not fully solved until March 2023, apart from some important but isolated results.

• Until very recently the only published result was that by Clifford Queen [30]. Queen's main result implies that when R^* is infinite + *some additional assumptions* on R hold, the elementary width of the group $\mathrm{SL}(2, R)$ is 5. As shown in [16] this implies, in particular, bounded elementary generation of all Chevalley groups $G(\Phi, R)$ under the same assumptions on R, with plausible bounds.

• The case of the groups over the usual polynomial ring $\mathbb{F}_q[t]$ long remained open. Only in 2018 has Bogdan Nica [28] established bounded elementary generation of $\mathrm{SL}(n, \mathbb{F}_q[t])$, $n \geq 3$. Next, in [16] we established bounded elementary generation of $\mathrm{Sp}(l, \mathbb{F}_q[t])$, $l \geq 2$, and

• The next breakthrough came in the preprints of Alexander Trost [38, 39] where he established bounded elementary generation of $\mathrm{SL}(n, R)$, for the ring of integers R of an arbitrary global function field K. First with a bound of the form $L(d, q) \cdot |\Phi|$, where the factor L depends on q and of the degree d of K, and then with the uniform bound. His method in [39] is similar to Morris' approach in [27].

1. Bounded generation of $G(\Phi, R)$

Combining the methods of [16] and [39], we are now able to come up with a complete solution in the general case. An important — and unexpected! — aspect of this work is the existence of *uniform* bounds. In the symplectic case this result is new even for the number case. All details are to be found in our forthcoming paper [18].

Theorem A. *Let Φ be a reduced irreducible root system of rank $l \geq 2$. Then there exists a constant $L = L(\Phi)$, depending on Φ alone, such that for any Dedekind ring of arithmetic type R, any element in $\mathrm{G}_{\mathrm{sc}}(\Phi, R)$ is a product of at most L elementary root unipotents.*

Roughly, the ingredients of the proof are as follows.

• For the **number case**, when R^* is infinite there is a definitive result by Morgan, Rapinchuk and Sury [25], with a small uniform bound $L \leq 9$, which can be improved in some cases.

Some uniform bound can be now easily derived by a version of the usual Tavgen's trick [37], Theorem 1, as described and generalised in [41, 33] and [16, 17].

• The uniform bound for $\mathrm{SL}(3, R)$ over imaginary quadratic rings was obtained by [27], see also [39]. Using the rank reduction methods based on Tavgen's lemma *and* stability, as in [16], we can reduce the analysis of $G(\Phi, R)$ for all *non-symplectic* root systems to $\mathrm{SL}(3, R)$.

• This leaves us with the analysis of $\mathrm{Sp}(2l, R)$, $l \geq 2$. What we haven't moticed when writing [16] is that bounded generation of $\mathrm{Sp}(2l, R)$, $l \geq 3$, also reduces to $\mathrm{SL}(3, R)$, with the help of the symplectic lemmas on switching long and short roots [16]. Thus, only $\mathrm{Sp}(4,R)$ requires separate analysis, since the bound given by Tavgen [37] is not uniform, it depends on the degree and discriminant of the number field K. However, in this case using our Sp_4-lemmas from [16] we are able to give a new proof in the style of [27].

• For the **function case**, $\mathrm{SL}(2,R)$ is not completely solved, so we have to rely on the reduction to rank two systems instead. Luckily, for $\mathrm{SL}(3,R)$ the uniform bound is given by Trost [39], which again (with the help of reduction lemmas from [16]) provides uniform bounds for all other groups of rank ≥ 2, with the sole exception of $\mathrm{Sp}(4,R)$. The key ingredient for this, bounded extraction of square roots from Mennicke symbols, is also contained in [39]. For this last case we succeed in imitating the proof from [27, 39] with our Sp_4-lemmas from [16].

2. Bounded generation of $\mathrm{St}(\Phi,R)$

Also, we obtained partial results towards bounded generation for the corresponding Steinberg groups. Again, we are interested in the bounded generation in terms of the set

$$X=\{x_\alpha(r)\mid r\in R,\ \alpha\in\Phi\}$$

o elementary generators.

However, this case turned out to be much more demanding. Apart from the bounded generation of the Chevalley groups themselves, it depends on the deep results on the finiteness of the (linear) K_2-functor, and on bunch of other difficult results of K-theory, such as stability theorem for K_2, centrality of K_2, etc.

Here is our second main result. So far we have been able to establish it only for the simply-laced systems.

Theorem B. *Let Φ be a reduced irreducible simply laced root system of rank ≥ 2, and let R be a Dedekind ring of arithmetic type. If $\Phi=\mathsf{A}_2$ assume additionally that R^* is infinite. Then* $\mathrm{St}(\Phi,\ R)$ *is boundedly elementary generated.*

The idea is to derive this result from Theorem A. It suffices to establish that the kernel $K_2(\Phi,R)$ of the projection $\mathrm{St}(\Phi,R)\longrightarrow G(\Phi,R)$ is finite and thus bounded elementary generation of $G(\Phi,R)$ implies that of $\mathrm{St}(\Phi,R)$. Here are the main sources on which we rely in this proof.

• The *stable* linear $K_2(R)$ is finite, for the function case this is proven by Hyman Bass and John Tate [3] and for the number case by Howard Garland [10]. (These finiteness results were generalised to higher K-theory by Daniel Quillen and Günter Harder, see the survey by Chuck Weibel [44]).

• However, we need similar results for the unstable K_2-functors $K_2(\Phi,R)$. For the *linear* case $\mathrm{SL}(n,R)$ there is a definitive stability theorem by Andrei Suslin and Marat Tulenbaev [36]. However, injective stability for Dedekind rings only starts with $n\geq 4$, so that for $\mathrm{SL}(3,R)$ one has to refer to van der Kallen [14] instead, which accounts for the extra-condition in this case.

• However, for other embeddings there are no stability theorems in the form we need them and starting where we want them to start. For instance, in the even orthogonal case the theorem of Ivan Panin [29] starts with $\mathrm{Spin}(10, R)$, whereas we would like to cover also $\mathrm{Spin}(8, R)$. In any case, there are no similar results for the exceptional embeddings.

Thus, we have to prove to prove a comparison theorem relating $K_2(\Phi, R)$ to $K_2(\mathrm{A}_3, R)$. This is accomplished by a combination of two techniques. On the one hand there are partial stability results for Dedekind rings developed by Hideya Matsumoto [24] and *surjective* stability of K_2 for some embeddings, established by Michael Stein [34] and one of us Eugene Plotkin. On the other hand, there are powerful recent calculations used to prove the centrality of K_2 for all Chevalley groups, by Andrei Lavrenov, Sergei Sinchuk, and Egor Voronetsky [19, 32, 20, 21, 42, 22].

• An essential obstacle in the symplectic case is that $K_2(\mathrm{C}_l, R)$ is the Milnor—Witt K_2^{MW}, rather than the usual Milnor K_2^M, as for all other cases (compare [35] for an explicit connection between $\mathrm{K}_2\mathrm{Sp}(R)$ and $\mathrm{K}_(R))$. As is well known, it may fail to be finite, which means that our approach does not work at all in this case. It does not mean that the result itself fails, but the proof would require an entirely different idea.

But even for non-symplectic multiply laced systems, where our approach could theoretically work, we were unable to overcome occurring technical difficulties related to the K_2-stability and comparison theorems. At least, as yet.

However, using specific calculations of $K_2(\Phi, \mathbb{F}_q[t])$ and $K_2(\Phi, \mathbb{F}_q[t, t^{-1}])$ by Eiichi Abe, Jun Morita, Jürgen Hurrelbrink and Ulf Rehmann [1, 11, 26, 31] we were able to establish similar results over $\mathbb{F}_q[t]$ and $\mathbb{F}_q[t,\, t^{-1}]$ also for the multiply laced systems, even the symplectic ones.

Theorem C. *Let Φ be a reduced irreducible root system, and $R = \mathbb{F}_q[t,\, t^{-1}]$ or $R = \mathbb{F}_q[t]$. In the latter case assume additionally that $\Phi \neq \mathsf{A}_1$. Then* $\mathrm{St}(\Phi,\, R)$ *is boundedly elementary generated.*

In the present talk we do not touch further closely related problems, such as commutator width or verbal width, or even relative versions of our results. Some indications and references can be found in [16, 17], more are coming in [18].

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