F-polynomials & Newton polytopes

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Abstract. We provide an effective algorithmic method for computation of Gross-Keel-Hacking-Kontsevich potential, F-polynomials and Bernstein-Kazhdan decoration function and its complexity bounds. For simply laced Lie algebras we make conjecture and provide experimental evidence for lattice points of Newton polytopes for Gross-Keel-Hacking-Kontsevich potential.

1. F-polynomials and Gross-Hacking-Keel-Kontsevich potentials

Let G be a group with the Lie algebra of simply-laced type , B_+ and B_- be its Borel subgroups, with the set of simple roots α_a, $a \in I$, W the Weyl group. The Gross-Hacking-Keel-Kontsevich potential (GHKK for short) W_{GHKK} is a function on the double Bruhat cell $G^{w_0,e} = B_- \cap B_+ \overline{w_0} B_+$, defined using cluster algebra's tools [5]. Because of validity the Fock-Goncharov conjecture in such cases [4], we get the polyhedral parametrization of canonical bases of the ring of regular functions on G/B arising from the tropicalizations of the potential.

Specifically, the ring of regular functions on the double Bruhat cell is endowed with the cluster algebra structure. Namely, for a reduced decomposition $\mathbf{i}$ of the longest element $w_0 \in W$ with length N, let $\Sigma_{\mathbf{i}}$ be a corresponding X-cluster seed and $Q_{\mathbf{i}}$ be the corresponding quiver (due to [1]). Then W_{GHKK} is a polynomial in the cluster variables $\Sigma_{\mathbf{i}}$([9]).

The frozen variables of this cluster algebra (and corresponding vertices of quiver) are labeled by the set $-I \cup I$. A seed Σ with underlying quiver Q is *optimal for a frozen vertex* $a \in -I \cup I$ if after deleting arrows between frozen vertices and a, the vertex a becomes a source of the quiver Q.

For the optimal seed Σ, the ath part of the GHKK-potential is equal to the value of the corresponding frozen cluster variable,

$$W_a = Y_a. \tag{1}$$

For a frozen $a \in I$, there exists an appropriate reduced word $\mathbf{i}'$, such that seed $\Sigma_{\mathbf{i}'}$ is optimal for a.

Because of that, for a given reduced decomposition $\mathbf{i}$, one can compute the half

$$W'_{GHKK} = \sum_{a\in I} W_a$$

of the GHKK-potential

$$W_{GHKK} = \sum_{a\in -I\cup I} W_a$$

using cluster mutations corresponding to 3-braid moves between the reduced decompositions of w_0 (for l and k, such that $a_{lk} = -1$, $s_k s_l s_k = s_l s_k s_l$). Namely, for computing W_a, we apply a sequence of cluster mutations corresponding to 3-braid moves which transform $\Sigma_{\mathbf{i}}$ into an optimal seeds for a, then W_a is the X-cluster variable at the frozen vertex labeled by a in the optimal seed computed in the variables of the seed $\Sigma_{\mathbf{i}}$. In variables of the seed $\Sigma_{\mathbf{i}}$, such an X-cluster variable is equal to the specification of the F-polynomial (see [3, 8]) and takes the form

$$W_a = Y_1^{c_{1a}(t)} \cdots Y_N^{c_{Na}(t)} \prod_i F_i(t)(Y_1, \cdots, Y_N)^{b_{ia}(t)}. \tag{2}$$

In the above formula we take notations of [8], where t means the end vertex of the path in the mutation graph from the optimal seed for a to $\Sigma_{\mathbf{i}}$ and Y_j's are cluster variables of $\Sigma_{\mathbf{i}}$.

Precisely (see [9]) W_a is of the form product of the frozen $Y_a(t)$ and an F-polynomial.

From [4] we can compute full GHKK-potential:

$$W_{GHKK} = W'_{GHKK} + \sum_{i\in I} Y_{i_s}^{-1}(1 + Y_{i_{s-1}}^{-1}(1 + Y_{i_{s-2}}^{-1}(1 + Y_{i_{s-3}}^{-1}(\cdots))))), \tag{3}$$

where $i_1, i_2, \dots i_s$ are indices of i in reduced decomposition $\mathbf{i}$.

2. Newton polytopes

We are interested of properties of the Newton polytopes of the individual terms W_a, $a \in I$, of the half-potential W_{GHHK}, as well as the Newton polytope of W_{GHHK}.

Fei in [2] conjectured that the Newton polytope of an F-polynomial has no interior integer points.

For minuscule weight $a \in I$, the validity of this conjecture for F-polynomials corresponding to terms W_a follows from Remark 5.17 [6]. Namely in such a case, the Newton polytope is a geometric realisation of a distributive lattice of the corresponding decorated graph DG_a. Since such a polytope is a convex hull of a subset of the vertices of a unit cube the claim follows.

We state the following

Conjecture 1. *For a simply-laced group G, and any reduced decomposition $\mathbf{i}$ of w_0, the Newton polytope of W'_{GHKK} is contains no interior integer lattice points.*

Conjecture 2. *For a simply-laced group G, and any reduced decomposition $\mathbf{i}$ of w_0, the Newton polytope of W_{GHKK} contains one interior integer lattice point.*

This conjectures are motivated by properties of affine Calabi-Yau manifolds and their mirror pairs should be affine too (thus rendering W'_{GHKK} void). Note that validity of this conjecture implies that the Newton polytopes of F-polynomials for frozens are void.

We made computer verification of Conjecture 1 for A_n, $n = 3,4,5,6,7$, D_n, $n = 4,5,6,7$, E_6, E_7 and Conjecture 2 for A_n, $n = 3,4,5$, D_4 and some cases of D_5.

Note that the straightforward cluster computation of W_{GHKK} is time consuming, because the division of Laurent polynomials in many variables is slow.

We use another approach. Namely, because of Theorem 1 in [4], we compute W_{GHKK} by applying the algorithm of [7] described in [11] for computing the Berenstein-Kazhdan decoration function Φ_{BK}.

Theorem 1. *For simply-laced G, and a given reduced decomposition* **i***, the Newton polytopes Φ_{BK} and W_{GHKK} are isomorphic under a unimodular transformation.*

Corollary 1. *The Newton polytopes Φ_{BK} contain one or zero interior integer lattice points if and only if the Newton polytopes W'_{GHKK} is contain one or zero interior integer lattice points.*

Thus, for the numeric verification of Conjectures we compute the Newton polytope Φ_{BK} using the algorithm [7] and Polymake.

We establish a bound on the complexity of the algorithm computing $\Delta_{w_0\Lambda_i,s_i\Lambda_i}$ and W_j with respect to the number of monomials in $\Delta_{w_0\Lambda_i,s_i\Lambda_i}$. Let K be the number of such monomials, and r be the rank of the Lie algebra.

Total complexity of generating the monomials of $\Delta_{w_0\Lambda_i,s_i\Lambda_i}$ is of complexity

$$O(r^4K) \sim O(r^2 * \textbf{length of string representation}).$$

W_j computation is bounded by multiplication complexity and number of edges in Gs: $O(r^2) * O(K * r^2) \sim O(r^4K)$. For a fixed r, this complexity is the lowest possible complexity being linear with respect to actual complexity to print out the answer.

Complexity of lattice point counting in polytopes has theoretical exponential upper bound with respect of number of inequalities defining polytope [10].

Actual computing speed is mostly determined by speed of Polymake operations. Average time of $\Delta_{w_0\Lambda_i,s_i\Lambda_i}$ and W_j computation for single simple root of D_6 with W'_{GHKK} polytope checking is around 2 seconds, for E_7 – 8 seconds. Checking Conjecture 2 for full W_{GHKK} polytope is much slower and can take from several hours to days even for D_5. For comparison, one computation for single simple root of D_6 without any Polymake operations takes around 70ms. For all computations we used a PC with dual 3.8 Ghz Intel® Xeon® Gold 5222 CPU running Ubuntu Linux.

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