Mathematics for non-mathematicians: memories of the future

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Abstract. The report discusses the problematic issues of modernization of mathematical education in higher education institutions on the basis of the experience of delivering training courses that use the capabilities of computer mathematics at the Department of Information Systems in Economics of St. Petersburg State University. The main result of the work performed is the development and justification of the author's approach to teaching mathematics to students, which makes it possible to combine key mathematical knowledge with calculations based on modern systems of symbolic computing and computer algebra. The implementation of the approach is described in the publicly available "Mathematica for a non-mathematician" textbook published under the auspices of the Moscow Center for Continuous Mathematical Education. The ideas of the approach have given the authors the victory in the competition of innovative educational projects of the Government of St. Petersburg. They formed the basis for a new project aimed at developing a domestic system of computer mathematics for science and education.

Background

2023 marks 20 years since the Department of Information Systems in Economics of St. Petersburg State University was established. For a number of years, it has been an administering department in the 'Applied Informatics in Economics' specialty. Starting from 2011, the department has been providing training of bachelors and masters in 'Business Informatics'. Graduating students of the department obtain the 'computational economist' qualification and have fundamental skills both in the field of computer science and mathematics and in the field of economics.

The mathematical education of business informaticians is an important component of the education plan. For a number of years of existence of the specialist program at the department, the 'Mathematics and computer' training course was taught to the students, where the key mathematical ideas were discussed by using the modern systems of symbolic computing and computer algebra. For those years, a number of textbooks have been published [3-11] which have accumulated the approaches developed by the authors and numerous problems the solving of which shows the breakthrough opportunities of computer mathematics both in computation and in visualization of the obtained results, compared with the traditional methods of teaching mathematical science.

As an example, Figure 1 represents building of the plot of function $\cos(x^2 - y^2)$ in the system Mathematica.

```
Plot3D[Cos[x^2 - y^2], {x, -Pi, Pi}, {y, -Pi, Pi},
 MeshStyle -> AbsoluteThickness[1],
 BoxStyle -> AbsoluteThickness[0.8],
 PlotPoints -> 60]
```

FIGURE 1. Building a graph by commands of Mathematica

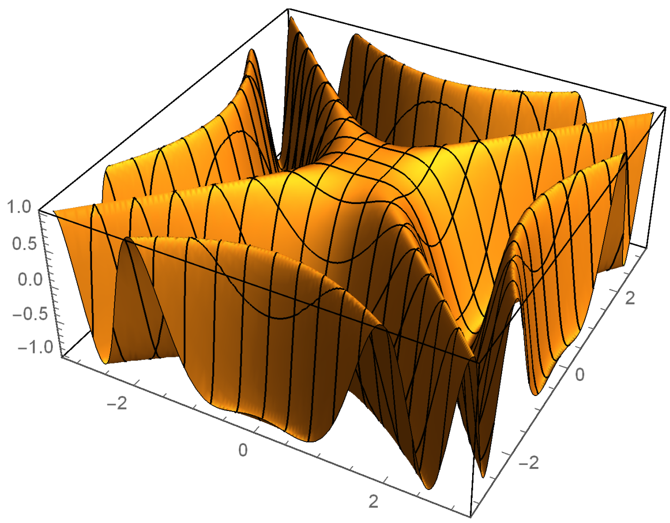

FIGURE 2. Command execution result

Regretfully, upon transition to the two-level system of higher education, there was no room for the "Mathematics and computer" course in the bachelor's degree program in 'Business Informatics'. However, the work performed for the specialist program in 2005-2011 has not fallen into oblivion[1] , and in 2018 the authors'

[1]Complete list of our books of that time is available at `https://web.archive.org/web/20160917081245/http://www.spbu-bi.ru/ru/science/publications.html`

application for the textbook 'Mathematics for non-mathematicians' was awarded a grant by the Vladimir Potanin Charitable Foundation which implements large-scale projects in the sphere of education and culture. Thanks to such support, the textbook [1] was created which manifests the opinion shared by the co-authors that mathematics may not be taught by drilling in routine operations which will never be applied by many students in their future life. The modern mathematical packages in skilled hands will solve equations and perform computations in a better way. The textbook has recalled to life the work of the early 2000s when the authors delivered lectures in 'Mathematics and computers' at the Department of Economics of St. Petersburg State University. The application prepared on the basis of the textbook received the Reward of St. Petersburg Government for winning the competition among innovative projects in the sphere of science and education (2021)[2].

The authors are sure that the developed textbook makes it possible for students of non-mathematical disciplines to get an insight into the opportunities provided by professional mathematical investigation tools which are a real alternative to the wide-spread office software. If published, the textbook will supplement the teaching-learning base of courses of quantitative methods for processing of economic information.

1. Key ideas of the textbook

- By using computer algebra systems, even today it is possible to conduct all computations which are standard for mathematics and its applications. All implications of this fact not only have not been recognized but even have not started to be considered seriously.
- The main general-purpose computer algebra systems are first of all programming languages of a very high level, near-living languages in their expressive power, and they should be learned exactly as languages but not as standard computer applications.
- Mathematicians are prone to underestimate the extent of dependence of the development of mathematics on the environment, firstly on computing facilities available. Even today the development of computer algebra has a dramatic impact on investigations in a lot of spheres of pure mathematics such as theory of groups, combinatorics, theory of numbers, commutative algebra, algebraic geometry, etc. In the near future, this influence will cover mathematics in its entirety and result in fundamental revision of the main areas of research, reappraisal of values, and a complete change in the mathematicians' work style.
- The furious antagonism caused among methodologists and many teachers of mathematics by the development of computer algebra is due to the fact that even in the nearest 10-15 years further development of such systems

[2] https://math-cs.spbu.ru/news/news-12-10-2021/

will result in complete devaluation of all traditional computing skills and the necessity of complete revision of teaching mathematics at the secondary and higher school levels.

- The furious antagonism caused among many representatives of Computer Science by the development of computer algebra is due to the fact that those systems also completely devaluate substantially all traditional programming skills. By using such systems, any informed amateur can write a program in a few minutes, while a similar program written in the algorithmic language Fortran or the currently popular dialects C and Python would require serious efforts of a professional programmer.

2. Prospects

The prospects for the development of mathematical education on the basis of use of systems of symbolic computing and computer algebra are described in the report 'The skies are falling [EVER MORE RAPIDLY]: mathematics for non-mathematicians' made at the joint seminar with Moscow State University 'Mathematics and informatics at secondary and higher school' held on March 9, 2023 (under the leadership of Academician A.L. Semenov and others). Presentation of the report and the video record of the workshop are available through the link. The recent events, including the unjustified sanctions of western vendors against the Russian Federation in relation to information and software products for science and education, have put a challenging problem of strategic security of domestic scientific research. The authors think that the creation of a competitive system of symbolic computing and computer algebra is real. We announced that by sending the project 'Computer mathematics: concepts of architectural, language and algorithmic support of computer algebra and quantum informatics systems' to the RSF competition of fundamental studies. The purpose of the project is to develop and implement new concepts of Computer Mathematics, namely: the basics of architectural, language and algorithmic support of computer algebra and quantum informatics systems for interface matching with languages, research style, and presentation of results, peculiar for modern mathematics. To achieve the purpose, it is planned to conduct large-scale comprehensive research in a number of inter-related areas, in particular:

- in the field of methodology: develop modern algorithms of symbolic computing and solve the critical difficulties of mathematical interpretation of their results, develop and practically promote the original methodology of using specialized mathematical packages for all levels of mathematical education in Russia, above all in training of researchers, mathematicians, and non-mathematicians;
- in the field of software: develop an image of a competitive mathematical package which would have the merits of modern foreign systems of computer

mathematics and implement the authors' experience of using such software tools;
- in the field of educational content: review, develop and design an educational content aimed at teaching substantiated and efficient application of modern computer facilities to various user categories;
- in the field of qualimetry and metrology: develop mathematical methods of valuation and forecasting of the statuses of Russian higher education institutions in focused ratings in mathematics and computer sciences in order to prepare and substantiate managerial decisions on improvement of their global competitiveness.

3. Scientific challenge to be solved by the project

Approximately 30 years ago, Doron Zeilberger stated that computers were becoming a thing of the same value for mathematics as telescopes and microscopes were for astronomy and biology in the XVII-th century. At the same time, speaking of the role of computers in mathematics, a lot of people confine themselves to the role of numerical calculations in applications on the one hand and to the formal derivation systems (automatic verification of theorems, verification of proofs, etc.) on the other. In those areas, especially in the first one, Russia has a fully developed school and major achievements. Meanwhile, computer mathematics is hardly limited to the above. In our opinion, in particular, systems of symbolic computing, especially computer algebra systems, will become far more significant both for mathematics itself and for its applications in the near future. In particular, for the recent years it has become clear that for a lot of real industrial projects not the applied mathematics and numerical methods are strongly sought for, but different branches of fundamental mathematics and advanced computer technology. Strong research groups are working in those areas, especially in Dubna, Moscow, and St. Petersburg; they have a vast experience of creating specialized packages focused on the performance of special types of computations for specific applications, usually in mathematics, physics, and astronomy, and partly for engineering applications. However, the functions implemented in such packages do not cover any wide branches of mathematics, and the packages themselves cannot be used directly in mathematical education. On top of that, quite often Russian mathematicians do not trust in the capabilities of symbolic computing systems, it is customary to point to "errors of systems of computer algebra" which, in our opinion, are absolutely imaginary and which result, on the one hand, from the failure to understand the basic principles of computer calculations and on the other hand, from objective difficulties of interpreting their results in traditional mathematical terms. It should be honestly acknowledged that the Russian mathematics in this respect is perceptibly in arrears of the world level. As concerns not specialized packages but full-fledged general-purpose computer algebra systems (general-purpose CAS), there are just a few of them in the world. Of course, there is a great number of extremely flexible and powerful specialized systems such as GAP, Magma, CoCoA, Singular, Pari, Lie and others, specially created for computations in specific fields such as theory of numbers, theory of groups, theory of representations, commutative algebra, algebraic geometry, etc. On the other hand, there is a great number of elementary systems, including very interesting ones, which are used at elementary steps of teaching mathematics, at the level of junior and secondary school. What practically does not exist is the intermediary step - systems of computer algebra which would cover a wide range of different branches of mathematics on an average-high level. If we set aside experimental, rudimentary and knowingly obsolete CAS, currently there are quite a few such modern systems, actually four: Maple, Mathematica, Axiom, and SageMath. This being said, Axiom, after the death of its author Richard Janks, has not been maintained for a long time, and SageMath is actually not an independent system but a convenient front-end which provides access to a few tens of specialized systems for a qualified user. Two of them, Mathematica and Maple, are commercial systems. They are absolutely remarkable, great software products which, when created in 1980s, were an outstanding achievement in computer mathematics and de-facto became a standard for organization of such systems. On the other hand, certain critical decisions related to their general architecture, computing, data structures, etc. which were taken at that time could not be changed subsequently exactly due to the commercial nature of such systems and the necessity of securing back compatibility. In addition, alterations made in the last versions of those systems are more and more focused not on the aspects which are important from the point of view of mathematics itself, but on various purely marketing issues: different specific extra-mathematical applications, computer graphics, etc. As opposed to Axiom, the both systems has no simple and natural language features for describing mathematical structures in terms of axioms or properties. Some exclusively important mathematical constructions (symbolic polynomials, symbolic matrices, etc.) were included therein only post-factum, with algorithms which were not most efficient. For the last 30-40 years, however, a great progress has occurred in the understanding of principles of computer mathematics. Currently it has become conceptually and technically possible to create systems with a language which in its vocabulary and expressive power is far closer to the human mathematicians' language. Such a language shall make it possible to describe mathematical structures in the manner actually used in mathematical books (with somewhat stricter syntax). This would enable implementing of a front-end of such systems on any national language. In addition, more efficient computing algorithms and methods were proposed in many scenarios which make it possible to perform calculations faster and by using smaller resources. In particular, parallel algorithm have substantially been worked out which were not used in traditional CAS. For the recent decades, the difficulties of translating the results of symbolic computing into the language of traditional mathematics have been recognized a lot better and overcome to a significant extent. This gives us faith to the real possibility of creating an up-to-date Russian system of symbolic computing with a front-end in the Russian language. Such a system could be vertically integrated and, on the one hand, available even for a schoolchild with respect to the requirements for the equipment and user qualification, and on the other hand, enabling quite sophisticated applications interesting for professional mathematics. It seems to us that none of the existing CAS, for all their undisputable advantages, satisfies such boundary conditions. The purpose of our project is to develop a theoretical framework of computer mathematics, to create a high-level Russian CAS with a full-fledged interface in the Russian and the English languages, to test the system on mathematical problems and to develop the basics for the use of the system in mathematical education. Apart from the merely scientific interest, the creation of such system would become the most important element of strategic security of scientific research and would be critical for mathematic education at very different levels. Such a system should preferably be an open-source system with clear separation of the kernel, the algorithm library supporting a variety of fields of modern pure and applied mathematics, with a developed data type system making it possible, on the language level, to build objects of new types by using the language structures most closely approximating the language of modern mathematics, as well as various interfaces allowing for modification of parts of the code by a qualified user. It is supposed to create a front-end software to ensure support of cloud computations, parallelization of algorithms, and interfaces for interaction with other computer algebra systems, e.g. Mathematica, Maple, Wolfram Alpha and others. We would keep in mind the availability of such a system for use at all levels of mathematical education in Russia and potentially in other countries, from secondary school to teaching professional mathematicians. The newest and the least technologically developed level, in a sense, would exactly be the medium level, i.e. teaching mathematics to non-mathematicians: both engineers, physicians, chemists, biologists and representatives of economic, social, and humanitarian disciplines. Historically, mathematics was extremely successful in a lot of applications, initially in astronomy and physics and then in other fields of natural science and engineering. Mathematics today could play the same role in all knowledge areas: biology, medicine, human science, social science, linguistics, cognitive science and others. If it is not so yet, this is only due to the fact that specialists in those fields are injured by the current modality of teaching mathematic starting from secondary school, do not know the mathematics they need, and which is worse, do not understand why they need such knowledge.

It is clear that the creation of a convenient and available system of computer algebra which the technical aspect of the matter could be delegated to, while concentrating on the conceptual aspect, could substantially resolve the problem. However, on the main obstacles is the lack of experience exactly of creation, debugging, and testing of large system of that type. Russian programmers have a great experience and top achievements in the field of writing short programs, competitive programming, etc. It seems that it is time to start creating a full-fledged high-level Russian CAS. The circumstances necessitate such a development.

We would like to note another important aspect of our project. There are still no functional quantum computers, in spite of numerous declarations made on this subject. Nevertheless, we are convinced that even today it is necessary to develop quantum algorithms of computer algebra and teach specialists in this area. We would note that the symbolic computing speed-up program proposed here is absolutely knew for this sphere, as only problems of numerical calculation speed-up were usually discussed therein. We would say that the existence of fast quantum algorithm makes it possible to consider post quantum computer algebra as a separate relevant field of research.

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[12] Internet links to the reports of the authors at scientific conferences, videos of training courses and the electronic edition of the textbook *Mathematica for non-mathematics*:

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