Fractional order differentiation of Meijer G-functions and their cases

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Abstract. We describe the Riemann-Liouville-Hadamard integro-differentiation of an arbitrary function to arbitrary symbolic order α which is realised in the Wolfram Language.

Introduction

The fractional derivative is a generalization of the mathematical concept of a derivative [1]. There are several different ways to generalize this concept, but the most of them coincide in corresponding classes of functions. When not only fractional, but also negative orders of the derivative are considered, the term differ-integral can be used.

We will use notation $\mathcal{D}_z^\alpha[f(z)]$ for Riemann-Liouville-Hadamard differ-integral for all $\alpha \in \mathbb{C}$. By definition of $\mathcal{D}_z^\alpha[f(z)]$ we put

$$\mathcal{D}_z^\alpha[f(z)] = \begin{cases} f(z), & \alpha = 0; \\ f^{(\alpha)}(z), & \alpha \in \mathbb{Z} \quad \text{and} \quad \alpha > 0; \\ \underbrace{\int_0^z dt \ldots \int_0^t dt \int_0^t}_{-\alpha\,\text{times}} f(t)dt, & \alpha \in \mathbb{Z} \quad \text{and} \quad \alpha < 0; \\ \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dz^n} \int_0^z \frac{f(t)dt}{(z-t)^{\alpha-n+1}}, & n = \lfloor \alpha \rfloor + 1 \quad \text{and} \quad \mathrm{Re}(\alpha) > 0; \\ \frac{1}{\Gamma(-\alpha)} \int_0^z \frac{f(t)dt}{(z-t)^{\alpha+1}}, & \mathrm{Re}(\alpha) < 0 \quad \text{and} \quad \alpha \notin \mathbb{Z}; \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(t)dt}{(z-t)^{\alpha}}, & \mathrm{Re}(\alpha) = 0 \quad \text{and} \quad \mathrm{Im}(\alpha) \neq 0, \end{cases} \tag{1}$$

where in the cases of divergence of integrals we use Hadamard finite part approach. Such construction is called *Riemann-Liouville-Hadamard fractional order derivative*. For "good enough" functions $f(z)$, provided convergence of above integrals at basic point $z=0$ it coincides with classical Riemann-Liouville definition, but for analytical functions can be extended to handle functions like $1/z$ or z^a or $(z^a)^b$ or e^{-z}/z or $\sqrt{z^2}/z$ or $\log(z^2)$ or $log(z)$ or $z^a\log^n(z)$, which are basic for building Taylor and Fourier series representations of more complicated functions like hypergeometric, Meijer G-function and Fox H-function.

For example, using (1), we obtain

$$\mathcal{D}_z^\alpha\left[\frac{1}{z}\right]=\begin{cases}(-1)^\alpha(1)_\alpha z^{-\alpha-1}, & \alpha\in\mathbb{Z},-1<\alpha;\\ \frac{z^{-\alpha-1}(-\psi(-\alpha)+\log(z)-\gamma)}{\Gamma(-\alpha)}, & \text{in other cases,}\end{cases}$$

$$\mathcal{D}_z^\alpha\left[z^\lambda\right]=\begin{cases}(-1)^\alpha(-\lambda)_\alpha z^{\lambda-\alpha}, & \alpha\in\mathbb{Z},\lambda\in\mathbb{Z},\lambda<0,\lambda<\alpha;\\ \frac{(-1)^{\lambda-1}z^{\lambda-\alpha}(\psi(-\lambda)-\psi(\lambda-\alpha+1)+\log(z))}{(-\lambda-1)!\Gamma(\lambda-\alpha+1)}, & \lambda\in\mathbb{Z},\lambda<0;\\ \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)}z^{\lambda-\alpha}, & \text{in other cases,}\end{cases}$$

$$\mathcal{D}_z^\alpha\left[\frac{\sqrt{z^2}}{z}\right]=\frac{z^{-\alpha-1}\sqrt{z^2}}{\Gamma(1-\alpha)}, \tag{2}$$

$$\mathcal{D}_z^\alpha\left[\log(z)\right]=\begin{cases}(-1)^{\alpha-1}(\alpha-1)!z^{-\alpha}, & \alpha\in\mathbb{Z},\alpha>0;\\ \frac{z^{-\alpha}(-\psi(1-\alpha)+\log(z)-\gamma)}{\Gamma(1-\alpha)}, & \text{in other cases,}\end{cases}$$

$$\mathcal{D}_z^\alpha\left[\frac{e^{-z}}{z}\right]=-\frac{z^{-\alpha}}{\Gamma(1-\alpha)}\,{}_2F_2(1,1;2,1-\alpha;-z)+$$

$$+z^{-\alpha-1}\begin{cases}(-1)^{-\alpha}(1)_\alpha, & \alpha\in\mathbb{Z}\wedge-1<\alpha;\\ \frac{\log(z)-\psi(-\alpha)-\gamma}{\Gamma(-\alpha)}, & \text{in other cases,}\end{cases}.$$

$$\mathcal{D}_z^\alpha\left[\log^2(z)\right]=\begin{cases}2(-1)^{\alpha-1}(\alpha-1)!z^{-\alpha}(-\psi(\alpha)+\log(z)-\gamma), & \alpha\in\mathbb{Z},\alpha>0;\\ \frac{z^{-\alpha}\left((\log(z)-\gamma)^2-\psi(1-\alpha)+\frac{\pi^2}{6}\right)-\psi'(1-\alpha)}{\Gamma(1-\alpha)}, & \text{in other cases.}\end{cases}$$

Here ψ is the digamma function, given by $\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$, ψ' gives the derivative of the digamma function.

Let us note that in (2) in the case $\lambda\in\mathbb{Z}$, $\lambda<0$ and $\lambda\geq\alpha$ the integration of z^λ can produce $\log(z)$. In this case we use Hadamard finite part. For example, when $\lambda=-1$ and $\alpha=-1$

$$\int\limits_\varepsilon^z\frac{dt}{t}=\log(z)-\log(\varepsilon)\Rightarrow\mathcal{D}_z^{-1}\left[z^{-1}\right]=f.p.\int\limits_0^z\frac{dt}{t}=\log(z).$$

We consider two approaches to calculating $\mathcal{D}_z^\alpha[f(z)]$. The first approach is to use Loran series expansions near zero. The second approach is to present function f through Meijer G-function and then find $\mathcal{D}_z^\alpha[f(z)]$ as a fractional derivative of this Meijer G-function.

1. Calculation of fractional derivatives and integrals by series expansion

Let consider the first approach to calculating $\mathcal{D}_z^\alpha[f(z)]$. Series expansion allows us to find $\mathcal{D}_z^\alpha[f(z)]$ because differ-integral applied to each term of Taylor series expansions of all functions near zero. So if

$$f(z) = z^b \sum_{n=0}^{\infty} c_n z^n \quad \Rightarrow \quad \mathcal{D}_z^\alpha[f(z)] = \sum_{n=0}^{\infty} c_n \mathcal{D}_z^\alpha[z^{b+n}]. \tag{3}$$

Sum representations by formula (3) we meet for functions like

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{2n+\nu},$$

but sometimes series expansions include $\log(z)$ function as in the logarithmic case of $K_0(z)$:

$$K_0(z) = -\left(\log\left(\frac{z}{2}\right) + \gamma\right) I_0(z) + \sum_{n=1}^{\infty} \frac{H_n}{(n!)^2} \left(\frac{z}{2}\right)^{2n},$$

with n-th harmonic number $H_n = \sum\limits_{k=1}^{n} \frac{1}{k}$ and γ is Euler–Mascheroni constant. It means, that we should consider more general series

$$f_L(z) = z^b \log^k(z) \sum_{n=0}^{\infty} c_n z^n,$$

and evaluate for arbitrary b, α and integer $k = 0, 1, 2, ...$ the following values

$$\mathcal{D}_z^\alpha[f_L(z)] = \sum_{n=0}^{\infty} c_n \mathcal{D}_z^\alpha \left[z^{b+n} \log^k(z)\right]. \tag{4}$$

So, in order to calculate (4) we should find $\mathcal{D}_z^\alpha\left[z^\lambda \log^k(z)\right]$ by the formula (1)

$$\mathcal{D}_z^\alpha\left[z^\lambda \log^k(z)\right] = \begin{cases} z^\lambda \log^k(z), & \alpha = 0; \\ (z^\lambda \log^k(z))^{(\alpha)}, & \alpha \in \mathbb{Z} \quad \text{and} \quad \alpha > 0; \\ \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dz^n} \int\limits_0^z \frac{t^\lambda \log^k(t) dt}{(z-t)^{\alpha-n+1}}, & n = \lfloor \alpha \rfloor + 1 \quad \text{and} \quad \mathrm{Re}(\alpha) > 0; \\ \frac{1}{\Gamma(-\alpha)} \int\limits_0^z \frac{t^\lambda \log^k(t) dt}{(z-t)^{\alpha+1}}, & \mathrm{Re}(\alpha) < 0; \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int\limits_0^z \frac{t^\lambda \log^k(t) dt}{(z-t)^\alpha}, & \mathrm{Re}(\alpha) = 0 \quad \text{and} \quad \mathrm{Im}(\alpha) \neq 0. \end{cases} \tag{5}$$

Here for $\alpha \in \mathbb{Z}$ and $\alpha > 0$

$$(z^\lambda \log^k(z))^{(\alpha)} = \sum_{j=0}^{\alpha} \binom{\alpha}{j} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-j+1)} z^{\lambda-j} \frac{d^{\alpha-j} \log^k(z)}{dz^{\alpha-j}}.$$

If some of integrals $\int\limits_0^z \frac{t^\lambda dt}{(z-t)^{\alpha+1}}$, $\int\limits_0^z \frac{t^\lambda dt}{(z-t)^{\alpha-n+1}}$, $\int\limits_0^z \frac{t^\lambda \log^k(t)dt}{(z-t)^{\alpha+1}}$, $\int\limits_0^z \frac{t^\lambda \log^k(t)dt}{(z-t)^{\alpha-n+1}}$ in (5) diverges we take Hadamard finite part of this integral.

2. The Meijer G-function and fractional calculus

It is known [2] (also see `ResourceFunction["MeijerGForm"]` in Wolfram Mathematica) that wide class of functions (hypergeometric type functions) can be defined as the functions, which generically can be represented through linear combinations of generalized Meijer G–function which is a very general special function of the form

$$G_{p,q}^{m,n}\left(z,r\,\middle|\, \begin{array}{c} a_1,...,a_n,a_{n+1},...,a_p \\ b_1,...,b_m,b_{m+1},...,b_q \end{array}\right) =$$

$$= \frac{r}{2\pi i}\int\limits_{\mathcal{L}} \frac{\prod\limits_{k=1}^{m}\Gamma(b_k+s)\prod\limits_{k=1}^{n}\Gamma(1-a_k-s)}{\prod\limits_{k=m+1}^{q}\Gamma(1-b_k-s)\prod\limits_{k=n+1}^{p}\Gamma(a_k+s)} z^{-\frac{s}{r}}\,ds, \tag{6}$$

where $r\in\mathbb{R}$, $r\neq 0$, $m\in\mathbb{Z}$, $m\geq 0$, $n\in\mathbb{Z}$, $n\geq 0$, $p\in\mathbb{Z}$, $p\geq 0$, $q\in\mathbb{Z}$, $q\geq 0$, $m\leq q$, $n\leq p$ (details about contour $\mathcal{L}$ separating "left" poles from "right" one see at `https://functions.wolfram.com/HypergeometricFunctions/MeijerG1/02/`).

Fractional order integral of this function with argument a z^r and parameter v can be described by the formula

$$\frac{1}{\Gamma(\alpha)}\int\limits_0^z (z-\tau)^{\alpha-1}\tau^{u-1}G_{p,q}^{m,n}\left(a\tau^r,\nu\,\middle|\, \begin{array}{c} a_1,...,a_n,a_{n+1},...,a_p \\ b_1,...,b_m,b_{m+1},...,b_q \end{array}\right)d\tau = z^{\alpha+u-1}\times$$

$$\times H_{p+1,q+1}^{m,n+1}\left(a^{1/\nu}z^{r/\nu}\,\middle|\, \begin{array}{c} \left(1-u,\frac{r}{\nu}\right),(a_1,1),...,(a_n,1),(a_{n+1},1),...,(a_p,1) \\ (b_1,1),...,(b_m,1),(b_{m+1},1),...,(b_q,1),\left(1-\alpha-u,\frac{r}{\nu}\right) \end{array}\right) \tag{7}$$

which is valid under corresponding conditions, providing convergence of above integral. Here $H_{p,q}^{m,n}$ is the Fox H-function defined by a Mellin–Barnes integral

$$H_{p,q}^{m,n}\left[z\,\middle|\, \begin{array}{cccc} (a_1,A_1) & (a_2,A_2) & \dots & (a_p,A_p) \\ (b_1,B_1) & (b_2,B_2) & \dots & (b_q,B_q) \end{array}\right] =$$

$$= \frac{1}{2\pi i}\int\limits_{\mathcal{L}} \frac{\prod\limits_{j=1}^{m}\Gamma(b_j+B_js)\prod\limits_{j=1}^{n}\Gamma(1-a_j-A_js)}{\prod\limits_{j=m+1}^{q}\Gamma(1-b_j-B_js)\prod\limits_{j=n+1}^{p}\Gamma(a_j+A_js)} z^{-s}\,ds,$$

where $\mathcal{L}$ is a certain contour separating the poles of the two groups of factors in the numerator. If the function $f(z)$ can be written as a finite sum of generalized Meijer G-function applying the formula (7) we can find fractional integral or derivative of $f(z)$ in the form of the Meijer G-function or Fox H-functions. Then we can write the Fox H-function as a simpler function if possible. Numerous examples of evaluation of fractional order integro derivatives users can find using `https://resources.wolframcloud.com/FunctionRepository/resources/FractionalOrderD` for example,

$$\mathcal{D}_z^{\alpha}[K_0(z)] = \frac{1}{2} G_{2,4}^{2,2}\left(\frac{z}{2}, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{1-\alpha}{2}, -\frac{\alpha}{2} \\ -\frac{\alpha}{2}, -\frac{\alpha}{2}, 0, \frac{1}{2} \end{matrix}\right).$$

Conclusion

Despite the fact that there are a large number of different approaches to fractional integro-derivation, for example, Riemann-Liouville, Caputo, Grünwald–Letnikov and others [1], these approaches are not so different. Indeed, calculating various fractional derivatives of a power function x^p, we almost always get the same result. In this paper we considered two approaches to calculating an arbitrary power of a differential operator $\frac{d}{dx}$ which are suitable for a wide class of functions.

References

[1] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional integrals and derivatives*, Amsterdam: Gordon and Breach Science Publishers, 1993.

[2] O. I. Marichev, *Handbook of Integral Transforms of Higher Transcendental functions (theory and algorithmic tables)*, Ellis Horwood Ltd, 1983.

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