Fractional order differentiation of Meijer G-functions and their cases

Oleg Marichev[1], Elina Shishkina[2]

[1] Wolfram Research Inc, Champaign, USA;
[2]Voronezh State University, Voronezh, Russia

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Introduction

Calculus teaches us how to compute derivatives of any integer order. We can interpret differentiation of negative integer order as a repeated integration. The zero order of differentiation gives the function itself. The question is how to generalize derivatives to non-integer order? The first attempt to discuss such an idea recorded in history was contained in the correspondence of Leibniz. In one of his letters to Leibniz, Bernoulli asked about the meaning of one theorem in the case of non-integer order of differentiation. Leibniz in his letters to L'Hôpital in 1695 and to Wallis in 1697 made some remarks on the possibility of considering differentials and derivatives of order 1/2 for to get acquainted with the main milestones of fractional calculus).

Introduction

Gamma function, generalization of the factorial function to nonintegral values,

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx, \qquad \mathrm{Re}(\alpha) > 0.$$

was introduced by the Swiss mathematician Leonhard Euler in the 18th century. As a result, it became possible to extend the formula

$$(x^p)^{(n)} = \frac{p!}{(p-n)!} x^{p-n}$$

for not-integer order n and the derivative of x^p of non-integer order α can be defined by

$$\frac{d^\alpha}{dx^\alpha} x^p = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} x^{p-\alpha}.$$

Introduction

The question of the $-1/2$ order derivative was raised by Euler, who, in 1738, mentioned that derivative of power function z^b has a meaning for non-integer order of differentiation. But only much later in 1820 Lacroix realized Euler's idea and presented an exact formula for the order $-1/2$ derivative of the power function z^b. Therefore, if function $f(x)$ is locally given by a convergent power series or $f(x)$ is an analytic function:

$$f(x) = \sum_{p=0}^{\infty} a_p x^p, \qquad a_p = \frac{f^{(p)}(0)}{p!},$$

then the derivative of order $\alpha > 0$ can be formally defined as

$$\frac{d^\alpha f(x)}{dx^\alpha} = \sum_{p=0}^{\infty} \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} a_p x^{p-\alpha}.$$

In 1823 first application of fractional calculus was discovered by Abel who was looking for a curve in the plane such that the time required for a particle to slide down the curve to its lowest point under the influence of gravity is independent of its initial position on the curve. Such curve is called the tautochrone. Abel found that for funding this curve it is necessary to solve the equation

$$\int_0^x \frac{f(t)dt}{(x-t)^{1/2}} = \varphi(x), \qquad x > 0.$$

It should be noticed that Abel solved more general equation

$$\int_0^x \frac{f(t)dt}{(x-t)^{\alpha}} = \varphi(x), \qquad x > 0,$$

where $0 < \alpha < 1$.

Introduction

Further, we note that in 1832 Liouville formally extended the formula for the integer derivative of the exponent $\frac{d^n}{dx^n}e^{bx}$ (b is a some number) to derivatives of an arbitrary order $\frac{d^\alpha}{dx^\alpha}e^{bx}$. Namely,

$$\frac{d^\alpha e^{bx}}{dx^\alpha} = b^\alpha e^{bx}.$$

Based on this formula, one can formally write the derivative of the order $\alpha \in \mathbb{R}$ of an arbitrary function f represented by the series

$$\frac{d^\alpha f(x)}{dx^\alpha} = \sum_{k=0}^{\infty} c_k b_k^\alpha e^{b_k x}, \qquad \text{where} \qquad f(x) = \sum_{k=0}^{\infty} c_k e^{b_k x}.$$

The limitation of this definition is related to the convergence of the series.

Formal definition of fractional integro-differentiation

So, starting from the 17th century, the need for a formal definition of fractional integro-differentiation gradually formed. We present here such definition following

- Ross, B., 1975. A brief history and exposition of the fundamental theory of fractional calculus. In: Ross B. (eds) Fractional Calculus and Its Applications. Lecture Notes in Mathematics, vol 457. Springer, Berlin, Heidelberg.

Operator $D^{\nu} f(z)$, $z \in \mathbb{C}$ is the integro-differential operator of order $\nu \in \mathbb{C}$ if and only if

1. If $f(z)$ is an analytic function of the complex variable z, the derivative $D^{\nu} f(z)$ is an analytic function of ν and z.
2. If $n \in \mathbb{N}$ then $D^n f(z)$ is an ordinary differentiation, $D^{-n} f(z)$ is an ordinary n-fold integration, $D^0 f(z) = f(z)$.
3. $D^{\nu}[af(z) + bg(z)] = aD^{\nu} f(z) + bD^{\nu} g(z)$.
4. $D^{\nu} D^{\mu} f(z) = D^{\nu+\mu} f(z)$.

Different approaches to classical fractional calculus

Operators

$$(I^{\alpha}_{a+}f)(x) = \frac{1}{\Gamma(\alpha)} \int\limits_a^x \frac{f(t)}{(x-t)^{1-\alpha}}dt, \qquad a < x \leq b,$$

$$(D^{\alpha}_{a+}f)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int\limits_a^x \frac{f(t)dt}{(x-t)^{\alpha-n+1}}, \qquad a < x \leq b.$$

are called left-sided **Riemann-Liouville** integral and derivative on the segment $[a, b]$. The definition of fractional integral is based on a generalization of the formula for an n-fold integral

$$\underbrace{\int\limits_a^x dx \dots \int\limits_a^x dx \int\limits_a^x}_{n} f(x)dx = \frac{1}{(n-1)!} \int\limits_a^x (x-t)^{n-1} f(t)dt, \qquad x > a.$$

Different approaches to classical fractional calculus

Now let consider how to generalize derivative of order n of the form

$$f^{(n)}(x) = \lim_{h \to 0} \frac{(\Delta_h^n f)(x)}{h^n}. \tag{1}$$

We get

$$f^{(\alpha)}(x) = \lim_{h \to +0} \frac{(\Delta_h^\alpha f)(x)}{h^\alpha}, \tag{2}$$

where

$$(\Delta_h^\alpha f)(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x-kh), \qquad \binom{\alpha}{k} = \frac{(-1)^{k-1}\alpha\Gamma(k-\alpha)}{\Gamma(1-\alpha)\Gamma(k+1)}. \tag{3}$$

For (2) is the left-hand sided **Grünwald–Letnikov** derivative ($\alpha > 0$) and integral ($\alpha < 0$).

Definition in Wolfram Mathematica

We will use notation $\mathcal{D}_z^{\alpha}[f(z)]$ for Riemann-Liouville-Hadamard differ-integral for all $\alpha \in \mathbb{C}$. By definition of $\mathcal{D}_z^{\alpha}[f(z)]$ we put

$$\mathcal{D}_z^{\alpha}[f(z)] = \begin{cases} f(z), & \alpha = 0; \\ f^{(\alpha)}(z), & \alpha \in \mathbb{Z} \quad \text{and} \quad \alpha > 0; \\ \underbrace{\int_0^z dt \ldots \int_0^t dt \int_0^t}_{-\alpha\,\mathrm{times}} f(t)dt, & \alpha \in \mathbb{Z} \quad \text{and} \quad \alpha < 0; \\ \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dz^n} \int_0^z \frac{f(t)dt}{(z-t)^{\alpha-n+1}}, & n = \lfloor \alpha \rfloor + 1 \quad \text{and} \quad \mathrm{Re}(\alpha) > 0; \\ \frac{1}{\Gamma(-\alpha)} \int_0^z \frac{f(t)dt}{(z-t)^{\alpha+1}}, & \mathrm{Re}(\alpha) < 0 \quad \text{and} \quad \alpha \notin \mathbb{Z}; \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(t)dt}{(z-t)^{\alpha}}, & \mathrm{Re}(\alpha) = 0 \quad \text{and} \quad \mathrm{Im}(\alpha) \neq 0, \end{cases} \tag{4}$$

In (4) in the cases of divergence of integrals we use Hadamard finite part approach. Such construction is called **Riemann-Liouville-Hadamard fractional order derivative**.

For "good enough" functions $f(z)$, provided convergence of above integrals at basic point $z = 0$ it coincides with classical Riemann-Liouville definition, but for analytical functions can be extended to handle functions like $1/z$ or z^a or $(z^a)^b$ or e^{-z}/z or $\sqrt{z^2}/z$ or $\log(z^2)$ or $log(z)$ or $z^a \log^n(z)$, which are basic for building Taylor and Fourier series representations of more complicated functions like hypergeometric, Meijer G-function and Fox H-function.

Examples

For example, using (4), we obtain

$$\mathcal{D}_z^{\alpha}\left[\frac{1}{z}\right] = \begin{cases} (-1)^{\alpha}(1)_{\alpha} z^{-\alpha-1}, & \alpha \in \mathbb{Z}, -1 < \alpha; \\ \frac{z^{-\alpha-1}(-\psi(-\alpha)+\log(z)-\gamma)}{\Gamma(-\alpha)}, & \text{in other cases,} \end{cases}$$

$$\mathcal{D}_z^{\alpha}\left[z^{\lambda}\right] = \begin{cases} (-1)^{\alpha}(-\lambda)_{\alpha} z^{\lambda-\alpha}, \qquad \alpha \in \mathbb{Z}, & \lambda \in \mathbb{Z}, \lambda < 0, \lambda < \alpha; \\ \frac{(-1)^{\lambda-1} z^{\lambda-\alpha}(\psi(-\lambda)-\psi(\lambda-\alpha+1)+\log(z))}{(-\lambda-1)!\Gamma(\lambda-\alpha+1)}, & \lambda \in \mathbb{Z}, \lambda < 0; \\ \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} z^{\lambda-\alpha}, & \text{in other cases,} \end{cases} \tag{5}$$

$$\mathcal{D}_z^{\alpha}\left[\frac{\sqrt{z^2}}{z}\right] = \frac{z^{-\alpha-1}\sqrt{z^2}}{\Gamma(1-\alpha)},$$

$$\mathcal{D}_z^{\alpha}[\log(z)] = \begin{cases} (-1)^{\alpha-1}(\alpha-1)! z^{-\alpha}, & \alpha \in \mathbb{Z}, \alpha > 0; \\ \frac{z^{-\alpha}(-\psi(1-\alpha)+\log(z)-\gamma)}{\Gamma(1-\alpha)}, & \text{in other cases.} \end{cases}$$

Examples

$$\mathcal{D}_z^\alpha\left[\frac{e^{-z}}{z}\right] = -\frac{z^{-\alpha}}{\Gamma(1-\alpha)}\,{}_2F_2(1,1;2,1-\alpha;-z)+$$

$$+z^{-\alpha-1}\begin{cases}(-1)^{-\alpha}(1)_\alpha, & \alpha\in\mathbb{Z},-1<\alpha;\\ \frac{\log(z)-\psi(-\alpha)-\gamma}{\Gamma(-\alpha)}, & \text{in other cases,}\end{cases}.$$

$$\mathcal{D}_z^\alpha\left[\log^2(z)\right] =$$

$$=\begin{cases}2(-1)^{\alpha-1}(\alpha-1)!z^{-\alpha}(-\psi(\alpha)+\log(z)-\gamma), & \alpha\in\mathbb{Z},\alpha>0;\\ \frac{z^{-\alpha}\left(\frac{\pi^2}{6}+(\log(z)-\psi(1-\alpha)-\gamma)^2-\psi'(1-\alpha)\right)}{\Gamma(1-\alpha)}. & \text{in other cases}\end{cases}$$

Here ψ is the digamma function, given by $\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$, ψ' gives the derivative of the digamma function.

Calculation of fractional derivatives and integrals by series expansion

Series expansion allows us to find $\mathcal{D}_z^\alpha[f(z)]$ because differ-integral applied to each term of power series expansions of all functions near zero. So if

$$f(z) = z^b \sum_{n=0}^{\infty} c_n z^n \quad \Rightarrow \quad \mathcal{D}_z^\alpha[f(z)] = \sum_{n=0}^{\infty} c_n \mathcal{D}_z^\alpha[z^{b+n}]. \qquad (6)$$

Sum representations by formula (6) we meet for functions like

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{2n+\nu}.$$

Calculation of fractional derivatives and integrals by series expansion

But sometimes series expansions include $\log(z)$ function as in the logarithmic case of $K_0(z)$:

$$K_0(z) = -\left(\log\left(\frac{z}{2}\right) + \gamma\right) I_0(z) + \sum_{n=1}^{\infty} \frac{H_n}{(n!)^2} \left(\frac{z}{2}\right)^{2n},$$

with n-th harmonic number $H_n = \sum_{k=1}^{n} \frac{1}{k}$ and γ is Euler–Mascheroni constant.

It means, that we should consider more general series $f_L(z) = z^b \log^k(z) \sum_{n=0}^{\infty} c_n z^n$, and evaluate for arbitrary b, α and integer $k = 0, 1, 2, \ldots$ the following values

$$\mathcal{D}_z^{\alpha}[f_L(z)] = \sum_{n=0}^{\infty} c_n \mathcal{D}_z^{\alpha}\left[z^{b+n} \log^k(z)\right]. \qquad (7)$$

Calculation of fractional derivatives and integrals by series expansion

In order to calculate (7) we should find $\mathcal{D}_z^\alpha\left[z^\lambda \log^k(z)\right]$ by (4)

$$\mathcal{D}_z^\alpha\left[z^\lambda \log^k(z)\right] = \begin{cases} z^\lambda \log^k(z), & \alpha = 0; \\ (z^\lambda \log^k(z))^{(\alpha)}, & \alpha \in \mathbb{Z} \quad \text{and} \quad \alpha > 0; \\ \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dz^n} \int\limits_0^z \frac{t^\lambda \log^k(t) dt}{(z-t)^{\alpha-n+1}}, & n = \lfloor \alpha \rfloor + 1, \mathrm{Re}(\alpha) > 0; \\ \frac{1}{\Gamma(-\alpha)} \int\limits_0^z \frac{t^\lambda \log^k(t) dt}{(z-t)^{\alpha+1}}, & \mathrm{Re}(\alpha) < 0; \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int\limits_0^z \frac{t^\lambda \log^k(t) dt}{(z-t)^{\alpha}}, & \mathrm{Re}(\alpha) = 0, \mathrm{Im}(\alpha) \neq 0. \end{cases} \tag{8}$$

Here for $\alpha \in \mathbb{Z}$ and $\alpha > 0$

$$(z^\lambda \log^k(z))^{(\alpha)} = \sum_{j=0}^{\alpha} \binom{\alpha}{j} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-j+1)} z^{\lambda-j} \frac{d^{\alpha-j} \log^k(z)}{dz^{\alpha-j}}.$$

Hadamard finite part of a singular integral

If some of integrals $\int\limits_0^z \frac{t^\lambda dt}{(z-t)^{\alpha+1}}$, $\int\limits_0^z \frac{t^\lambda dt}{(z-t)^{\alpha-n+1}}$, $\int\limits_0^z \frac{t^\lambda \log^k(t) dt}{(z-t)^{\alpha+1}}$, $\int\limits_0^z \frac{t^\lambda \log^k(t) dt}{(z-t)^{\alpha-n+1}}$ in (8) diverges we take Hadamard finite part of this integral.

The concept of the "finite part" of a singular integral introduced by Hadamard based on dropping some divergent terms and keeping the finite part.

Hadamard finite part of a singular integral

Let a function $f = f(x)$ be integrable in an interval $\varepsilon < x < A$ for any $0 < \varepsilon$, $\varepsilon < A < \infty$ and the representation

$$\int\limits_{\varepsilon<x<A} f(x)\,dx = \sum_{k=1}^{N} a_k \varepsilon^{-\lambda_k} + h \ln\frac{1}{\varepsilon} + J_\varepsilon \tag{9}$$

hold valid, where a_k, h, λ_k are some constant positive numbers independent of A. If the limit $\lim\limits_{\varepsilon\to 0} J_\varepsilon$ exists, then it is called the Hadamard finite part of the singular integral of the function f. The function $f = f(x)$ is said to possess the Hadamard property at the origin. The standard notation for the finite part of the Hadamard singular integral is as follows

$$f.p. \int\limits_{x<A} f(x)\,dx = \lim_{\varepsilon\to 0} J_\varepsilon. \tag{10}$$

Hadamard finite part of a singular integral

Example. For example, when in $\int_0^z \frac{t^\lambda \log^k(t)dt}{(z-t)^{\alpha-n+1}}$ we have $k=0$, $\lambda=-1$ and $\alpha=-1$

$$\int_\varepsilon^z \frac{dt}{t} = \log(z) - \log(\varepsilon)$$

then

$$\mathcal{D}_z^{-1}\left[z^{-1}\right] = f.p.\int_0^z \frac{dt}{t} = \log(z).$$

The Meijer G-function and fractional calculus

It is known

- O. I. Marichev, *Handbook of Integral Transforms of Higher Transcendental functions (theory and algorithmic tables)*, Ellis Horwood Ltd, 1983.
- `ResourceFunction["MeijerGForm"]` in Wolfram Mathematica

that wide class of functions (hypergeometric type functions) can be defined as the functions, which generically can be represented through linear combinations of generalized Meijer G–function.

The Meijer G-function and fractional calculus

The Meijer G-function is a very general special function of the form

$$G_{p,q}^{m,n}\left(z, r \,\middle|\, \begin{array}{c} a_1, ..., a_n, a_{n+1}, ..., a_p \\ b_1, ..., b_m, b_{m+1}, ..., b_q \end{array}\right) =$$

$$= \frac{1}{2\pi i} \int\limits_{\mathcal{L}} \frac{\prod\limits_{k=1}^{m} \Gamma(b_k + s) \prod\limits_{k=1}^{n} \Gamma(1 - a_k - s)}{\prod\limits_{k=m+1}^{q} \Gamma(1 - b_k - s) \prod\limits_{k=n+1}^{p} \Gamma(a_k + s)} z^{-\frac{s}{r}} \, ds, \qquad (11)$$

where $r \in \mathbb{R}$, $r \neq 0$, $m \in \mathbb{Z}$, $m \geq 0$, $n \in \mathbb{Z}$, $n \geq 0$, $p \in \mathbb{Z}$, $p \geq 0$, $q \in \mathbb{Z}$, $q \geq 0$, $m \leq q$, $n \leq p$ (details about contour $\mathcal{L}$ separating "left" poles from "right" one see at `https://functions.wolfram.com/HypergeometricFunctions/MeijerG1/02/`.

Fractional order integral of this function with argument a z^r and parameter v can be described by the formula

$$\frac{1}{\Gamma(\alpha)}\int\limits_0^z (z-\tau)^{\alpha-1}\tau^{u-1} G_{p,q}^{m,n}\left(a\tau^r,\nu \,\middle|\, \begin{matrix} a_1,\ldots,a_n,a_{n+1},\ldots,a_p \\ b_1,\ldots,b_m,b_{m+1},\ldots,b_q \end{matrix}\right) d\tau = z^{\alpha+u-1}\times$$

$$\times H_{p+1,q+1}^{m,n+1}\left(a^{\frac{1}{\nu}} z^{\frac{r}{\nu}} \,\middle|\, \begin{matrix} \left(1-u,\frac{r}{\nu}\right),(a_1,1),\ldots,(a_n,1),(a_{n+1},1),\ldots,(a_p,1) \\ (b_1,1),\ldots,(b_m,1),(b_{m+1},1),\ldots,(b_q,1),\left(1-\alpha-u,\frac{r}{\nu}\right) \end{matrix}\right)$$

which is valid under corresponding conditions, providing convergence of above integral.

The Meijer G-function and fractional calculus

Here $H_{p,q}^{m,n}$ is the Fox H-function defined by a Mellin–Barnes integral

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,A_1) & (a_2,A_2) & \dots & (a_p,A_p)\\(b_1,B_1) & (b_2,B_2) & \dots & (b_q,B_q)\end{matrix}\right.\right] =$$

$$= \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\prod_{j=1}^{m}\Gamma(b_j+B_js)\ \prod_{j=1}^{n}\Gamma(1-a_j-A_js)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-B_js)\ \prod_{j=n+1}^{p}\Gamma(a_j+A_js)} z^{-s}\, ds,$$

where $\mathcal{L}$ is a certain contour separating the poles of the two groups of factors in the numerator. If the function $f(z)$ can be written as a finite sum of generalized Meijer G-function we can find fractional integral or derivative of $f(z)$ in the form of the Meijer G-function or Fox H-functions.

The Meijer G-function and fractional calculus

Then we can write the Fox H-function as a simpler function if possible. Numerous examples of evaluation of fractional order integro derivatives users can find using
`https://resources.wolframcloud.com/FunctionRepository/resources/FractionalOrderD` for example,

$$\mathcal{D}_z^{\alpha}[K_0(z)] = \frac{1}{2} G_{2,4}^{2,2}\left(\frac{z}{2}, \frac{1}{2} \,\middle|\, \begin{array}{c} \frac{1-\alpha}{2}, -\frac{\alpha}{2} \\ -\frac{\alpha}{2}, -\frac{\alpha}{2}, 0, \frac{1}{2} \end{array} \right).$$

Example

Let consider $\frac{e^{-z}}{z^n}$, $n \in \mathbb{N}$. Using `MeijerGForm` from Wolfram Mathematica we can write

$$\frac{e^{-z}}{z^n} = G^{1,0}_{0,1}\left(z \,\middle|\, \begin{matrix} - \\ -n \end{matrix}\right) = \frac{1}{2\pi i}\int_{\mathcal{L}} \Gamma(s-n) z^{-s} ds, \qquad \operatorname{Re} s > n.$$

Expanding the exponent in a series, we get

$$\frac{e^{-z}}{z^n} = z^{-n}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}z^k = \sum_{k=n}^{\infty}\frac{(-1)^k}{k!}z^{k-n} + \sum_{k=0}^{n-1}\frac{(-1)^k}{k!}z^{k-n} =$$

$$= \frac{e^{-z} - \sum\limits_{k=0}^{n-1}\frac{(-1)^k}{k!}z^k}{z^n} + \sum_{k=0}^{n-1}\frac{(-1)^k}{k!}z^{k-n}.$$

Example

Using formula (8) we obtain for $n \in \mathbb{N}$

$$\mathcal{D}_z^{\alpha}\left[\frac{e^{-z}}{z^n}\right] = \frac{(-1)^n z^{-\alpha}}{\Gamma(n+1)\Gamma(1-\alpha)}\, {}_2F_2(1,1;n+1,1-\alpha;-z)+$$

$$+\, z^{k-\alpha-n}\sum_{k=0}^{n-1}\frac{(-1)^k}{k!}\begin{cases} (-1)^{\alpha}(n-k)_{\alpha},\ \alpha\in\mathbb{Z}, k-n<\alpha \\ \frac{(-1)^{k-n-1}(\log(z)+\psi(n-k)-\psi(k-n-\alpha+1))}{(-k+n-1)!\Gamma(k-n-\alpha+1)}, \\ \text{in other cases.}\end{cases} \tag{12}$$

When n is not natural number we get

$$\mathcal{D}_z^{\alpha}\left[\frac{e^{-z}}{z^n}\right] = \frac{\Gamma(1-n)}{\Gamma(1-n-\alpha)} z^{-\alpha-n}\, {}_1F_1(1-n;1-n-\alpha;-z).$$

Example

Here ${}_pF_q$ is the *generalized hypergeometric function* is defined as a power series

$$
{}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}.
$$

Example

We can verify formulas using `Integrate` with option `GenerateConditions->False`. For example when $n = 1$:

$$\frac{1}{\Gamma(-\alpha)} \int_0^z (z-t)^{-\alpha-1} \frac{e^{-t}}{t} dt =$$

$=$`Integrate[`$\frac{(z-t)^{-\alpha-1}}{\Gamma(-\alpha)} \frac{e^{-t}}{t}$`,t, 0, z, GenerateConditions -> False]`$=$

$$= \frac{z^{-\alpha-1}\left(z\,{}_2F_2(1,1;2,1-\alpha;-z) - \alpha H_{-\alpha-1} + \alpha \log(z)\right)}{\alpha\Gamma(-\alpha)},$$

where $H_s = \gamma + \psi(s+1)$ is the harmonic number ($H_n = \sum\limits_{k=1}^{n} \frac{1}{k}$ for integer n), $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ is the digamma function, γ is Euler–Mascheroni constant.

Example

So, we can calculate numerically $\mathcal{D}_z^{\alpha}\left[\frac{e^{-z}}{z}\right]$ and $\frac{1}{\Gamma(-\alpha)}\int\limits_0^z (z-t)^{-\alpha-1}\frac{e^{-t}}{t}dt$.

We get for `z -> Random[Complex]`, α`-> -2`:

$$\mathcal{D}_z^{\alpha}\left[\frac{e^{-z}}{z}\right] \to -0.400227 - 0.224905i,$$

$$\frac{1}{\Gamma(-\alpha)}\int\limits_0^z (z-t)^{-\alpha-1}\frac{e^{-t}}{t}dt \to -0.400227 - 0.224905i.$$

This result coincides with right side of the first formula (12) for $n=1$.

THANK YOU FOR ATTENTION.