The (discrete) Heisenberg group and (restricted) Young's lattices

Andrei Malyutin

Abstract. We discuss some aspects of the problem of describing the absolute of the discrete Heisenberg group, focusing on the connections of this problem with the asymptotic behavior of (restricted) Young's lattices.

In [2–9], the theory of the absolute of graphs, groups, and semigroups is developed. The absolute is a probabilistic-topological boundary of a group (semigroup, graph) generated by random walks: the absolute of a group with a fixed set of generators can be defined as the set of ergodic Markov measures (i. e., random walks on the group) having the same system of cotransition probabilities as the simple (right) random walk generated by the uniform distribution on the generators. By now, a description of the absolute has been obtained for the free group and some other hyperbolic groups, for Abelian groups and semigroups, and for the discrete Heisenberg group.

In the theory of absolute, a prominent role is played by the functions describing the number of paths of given length between vertices of a graph (the Cayley graph of a group) and various relations between these functions. In the Cayley graph of the discrete Heisenberg group with the standard set of generators, the number of geodesic paths between two given points is equal to the number of restricted partitions with some condition on the size and the number of parts, or, which is the same, the number of Young diagrams of given area fitting in a rectangle of given size.

In particular, in order to describe the absolute of the discrete Heisenberg group, we use the following lemma (see [9, Lemma 1]). Let $P_{w,h}(N)$ denote the number of Young diagrams of area N fitting in a rectangle of width w and height h. Then

$$\frac{P_{w,h}(N+1)}{P_{w,h}(N)} \xrightarrow[w,h,N\in\mathbb{N}]{\min\{w,h,N,wh-N\}\to+\infty} 1.$$

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When analyzing whether the absolute of the discrete Heisenberg group is compact, the following conjecture arises.

$$\left|\frac{P_{w,h}(N)}{P_{w,h+1}(N)} - \frac{P_{w,h+1}(N)}{P_{w,h+2}(N)}\right| \xrightarrow[w,h,N\in\mathbb{N}]{\min\{w,h,N,wh-N\}\to+\infty} 0.$$

We will discuss the following related asymptotics. Let $P_w(N)$ be the number of Young diagrams of area N and of width $\leq w$. Then

$$\left|\frac{P_w(N)}{P_w(N+w)} - \frac{P_w(N+w)}{P_m(N+2w)}\right| \xrightarrow[w,N\in\mathbb{N}]{w+N\to+\infty} 0,$$

$$\left|\frac{P_w(N)}{P_{w+1}(N)} - \frac{P_{w+1}(N)}{P_{w+2}(N)}\right| \xrightarrow[w,N\in\mathbb{N}]{w+N\to+\infty} 0.$$

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Andrei Malyutin
St. Petersburg Department of
Steklov Institute of Mathematics
St. Petersburg State University
St. Petersburg, Russia
e-mail: malyutin@pdmi.ras.ru