A Normal Form of Derivations for Quantifier-Free Sequent Calculi With Nonlogical Axioms

Alexander Sakharov

1. Introduction

The core of AI systems is domain knowledge that is commonly expressed as logical programs or knowledge base rules. A great deal of research has been devoted to logical characterizations of these systems. These characterizations give formal descriptions of otherwise obscure systems and make their results explainable. AI systems are regularly categorized by various non-standard calculi. Models may not be available for these calculi. In this situation, the task of developing an inference method for a particular non-standard logic is often a complicated research project.

Arguably, sequents are the most common notation in the specification of proof theories. Sequent calculi have been used in formalizations of a variety of logics. Sequent calculi are suitable for establishing derivation properties. Nonetheless, they do not facilitate inference methods because some of their axioms and inference rules constitute infinite branching points in derivation search and because of the variety of rule choices at any derivation step.

We suggest sequent calculi with inference rules of certain forms as a framework for representing standard and non-standard logics having AI utility. Logical rules in these quantifier-free calculi are introduction rules for logical connectives. Domain knowledge is expressed by nonlogical axioms in the form of sequents in these calculi. Inference in these calculi can be restricted to a normal form for which a variant of the subformula property holds.

2. Sequent Calculi

We consider quantifier-free languages because typical AI languages such as logic programming and knowledge base languages exclude quantifiers [6]. Skolem functions serve as an alternative to quantifiers. Quantifiers are problematic for some non-standard logics. As usual, formulas are built recursively from atoms and logical connectives, atom arguments are terms built recursively from object variables, constants, and functions. We limit connectives to unary and binary. The languages of particular calculi could be more restricted. A number of calculi related to AI are propositional - they do not include variables. Datalog [6] does not include functions.

Sequent calculi have axioms and inference rules [4]. Inference rules have one or more premises and one conclusion, each of them is a sequent. We limit the number of premises to two. Axioms are basically inference rules with zero premises. Sequent calculi include logical axiom $A \vdash A$ or a similar one. Upper-case Latin letters denote formulas in inference rules. Upper-case Greek letters are metavariables denoting formula multisets.

It is known that sequent calculi do not necessarily have an adequate expressiveness for some intricate logics. Sequent calculus extensions such as hypersequents [10] have been developed to address these unusual cases. These different extensions are not covered in this paper. It is fair to say that it is not realistic to have a universal language whose expressive power is sufficient for a variety of sequent calculus extensions. Allowing additional logical axioms makes it possible to express complicated logics as ordinary sequent calculi, but these axioms may compromise important properties of sequent calculi.

A substitution is a finite mapping of object variables to terms. The result of applying a substitution θ to a formula A is the expression $A\theta$ obtained from A by simultaneously replacing every occurrence of every variable from θ by the term with which the variable is associated. $A\theta$ is called an instance of A. The notions of substitution and instance can be extended onto sequents.

Nonlogical axioms are sequents containing formulas, no multiset or formula metavariables occur in them. Any nonlogical axiom with variables represents infinitely many sequents. Each of these sequents is an instance of the axiom. Nonlogical axioms represent knowledge base rules and facts (or logical programs). They express properties of concrete predicates and functions.

Usually, the outcome of inference is sequents $\vdash G$ where formula G is called a goal. Unlike goals for theorem provers, goals for AI systems as well as formulas in nonlogical axioms are shallow formulas. An axiom is called reducible if it has an instance with two or more identical formulas. A calculus is called consistent if sequent $\vdash$ is not derivable. Inconsistent calculi without nonlogical axioms are not worth investigating but it is acceptable for nonlogical axioms to be the source of inconsistency. Argumentation deals with inconsistent sets of nonlogical axioms [1].

Inference rules in sequent calculi are split into structural and logical. The structural rules are essentially universal for all of the calculi whereas logical rules vary. Given the multiset view of antecedents and succedents, the structural rules are weakening, contraction, and cut [4]. Some of these structural rules may be missing in some calculi. Some calculi restrict the number of formulas in succedents to one. We do not consider calculi with other constraints.

Every formula from the conclusion of a logical inference rule that is not identical to a formula from a premise is called principal. Every formula from premises that is not identical to a formula from the conclusion is called active.

All other formulas are called contexts. Let $\diamond$ denote a unary connective, $\circ$ denote a binary connective. Let $\diamond\Pi$ denote $\{\diamond A | A \in \Pi\}$.

Definition 1. *A logical inference rule is called an introduction rule if it has one of the following forms and does not have any applicability provisos.*

$$\frac{A, \Gamma \vdash \Pi}{\diamond A, \Gamma \vdash \Pi}\ L1 \qquad \frac{A \vdash \diamond\Pi}{\diamond A \vdash \diamond\Pi}\ LP \qquad \frac{\Gamma \vdash A, \Pi}{\Gamma \vdash \diamond A, \Pi}\ R1 \qquad \frac{\diamond\Gamma \vdash A}{\diamond\Gamma \vdash \diamond A}\ RP$$

$$\frac{A, \Gamma \vdash \Pi}{\Gamma \vdash \diamond A, \Pi}\ F1 \qquad \frac{\Gamma \vdash A, \Pi}{\diamond A, \Gamma \vdash \Pi}\ B1$$

$$\frac{\Gamma \vdash}{\diamond\Gamma \vdash}\ LO \qquad \frac{\Gamma \vdash A}{\diamond\Gamma \vdash \diamond A}\ RL \qquad \frac{\vdash \Pi}{\vdash \diamond\Pi}\ RO \qquad \frac{A \vdash \Pi}{\diamond A \vdash \diamond\Pi}\ LR$$

$$\frac{A, B, \Gamma \vdash \Pi}{A \circ B, \Gamma \vdash \Pi}\ L2 \qquad \frac{\Gamma \vdash A, B, \Pi}{\Gamma \vdash A \circ B, \Pi}\ R2 \qquad \frac{A, \Gamma \vdash B, \Pi}{\Gamma \vdash A \circ B, \Pi}\ F2 \qquad \frac{A, \Gamma \vdash B, \Pi}{A \circ B, \Gamma \vdash \Pi}\ B2$$

$$\frac{A, \Gamma \vdash \Pi \quad B, \Gamma \vdash \Pi}{A \circ B, \Gamma \vdash \Pi}\ LA \qquad \frac{B, \Gamma \vdash \Pi \quad B, \Delta \vdash \Sigma}{A \circ B, \Gamma, \Delta \vdash \Pi, \Sigma}\ LM$$

$$\frac{\Gamma \vdash A, \Pi \quad \Gamma \vdash B, \Pi}{\Gamma \vdash A \circ B, \Pi}\ RA \qquad \frac{\Gamma \vdash A, \Pi \quad \Delta \vdash B, \Sigma}{\Gamma, \Delta \vdash A \circ B, \Pi, \Sigma}\ RM$$

$$\frac{B, \Gamma \vdash \Pi \quad \Delta \vdash A, \Sigma}{\Gamma, \Delta \vdash A \circ B, \Pi, \Sigma}\ FM \qquad \frac{B, \Gamma \vdash \Pi \quad \Delta \vdash A, \Sigma}{A \circ B, \Gamma, \Delta \vdash \Pi, \Sigma}\ BM$$

Definition 2. *A sequent calculus is called a L_A calculus if it has one logical axiom $A \vdash A$ and possibly nonlogical axioms, the cut rule, possibly the two weakening rules, possibly the two contraction rules, some introduction logical rules, and*
- *for every unary connective $\diamond$, the rules with this connective are limited to one R1 rule and possibly one L1 or LP rule, one RP rule and possibly one L1 rule, one F1 rule and possibly one B1 rule, one RL rule and one of LO/L1 rules, or one LR rule and one of RO/R1 rules,*
- *for every binary connective $\circ$, the rules with this connective are limited to one R2 rule and possibly one LA rule, one R2 rule and possibly one LM rule, one RA rule and possibly one L2 rule, one RM rule and possibly one L2 rule, one F2 rule and possibly one BM rule, or one FM rule and possibly one B2 rule.*

The idea of introduction rules is that every formula from a premise is a subformula of some formula from the conclusion. There are some non-standard logics that cannot be specified by calculi with introduction rules. One example of that is temporal logics. Their sequent calculi include logical rules in which some formulas in the conclusion are proper subformulas of ones in the premise [5]. The choice of the introduction rule forms is dictated by the desideratum of the subformula property. For that reason, rules with syntactic constraints on both antecedent and succedent contexts such as $S5$ rules are excluded [10]. No surprise that the introduction rules correspond to the calculi that enjoy cut admissibility in the absence of nonlogical axioms.

Clearly, the quantifier-free fragments of classical and intuitionistic first-order logics are L_A calculi. Other examples of L_A calculi include multiplicative linear logic [2], the LK_{-c} calculi of evaluable non-Horn knowledge bases [7], modal logic $S4$ [10], standard deontic logic [1].

3. Normal Form

The object of this investigation is sets (families) of L_A calculi in which structural and logical inference rules are fixed and every calculus in the set has its own set of nonlogical axioms. Any calculus in a set corresponds to an AI system, predicates and functions are constants from a finite set determined by the domain of the system. Basically, such set of calculi corresponds to a logic for a variety of domains. We do not consider calculi without the cut rule. This rule plays the role of Modus Pones. Without Modus Pones, nonlogical axioms are useless.

Let $[\Gamma]$ denote the result of applying zero or more possible contractions to multiset Γ. If a calculus set does not include contraction, then $[\Gamma] = \Gamma$. If a calculus includes both weakening and contraction, then the [ ] operation eliminates all duplicate formulas. If a calculus includes contraction and does not include weakening, then this operation is non-deterministic. Let us modify the conclusion of the cut rule and all logical rules by applying [ ] to both the antecedent and the succedent. For instance, cut and BM become

$$\frac{\Gamma \vdash A, \Delta \quad A, \Pi \vdash \Sigma}{[\Gamma, \Pi] \vdash [\Delta, \Sigma]}\ cut \qquad\qquad \frac{A, \Gamma \vdash \Pi \quad \Delta \vdash B, \Sigma}{[A \circ B, \Gamma, \Delta] \vdash [\Pi, \Sigma]}\ BM$$

Definition 3. *The calculi obtained from L_A by applying [ ] to both antecedent and succedent in the conclusion of cut and logical inference rules are called L'_A.*

Proposition 1. *For any L_A calculus and its L'_A counterpart, any L_A derivation can be transformed into a L'_A derivation with the same endsequent and vice versa.*

Proposition 2. *The contraction rules are admissible in L'_A derivations for calculi with non-reducible nonlogical axioms.*

Theorem 1. *(normal form) For a consistent L'_A calculus with non-reducible nonlogical axioms, every derivation with endsequent $\vdash G$ can be transformed into such derivation with the same endsequent and without contractions that the following holds:*

1) (weak subformula property) Every formula in the derivation is G, its subformula, or an instance of a formula from a nonlogical axiom or its subformula.

2) Every cut formula is an instance of a formula from a nonlogical axiom.

3) If one premise of cut is the conclusion of a logical rule, then the cut formula is principal in the logical rule and the other premise is a nonlogical axiom or the conclusion of another cut.

4) The conclusion of every weakening is the premise of $L2, R2, F2, B2, LA, RA$, or another weakening.

5) Every weakening formula is active in the first descendant $L2, R2, F2, B2$ rule or adds a formula to the context of a premise of the first descendant LA, RA rule from the context of the other premise of the latter rule.

Theorem 2. *For a L'_A calculus without LP, RP rules and a simplification order* [3], *every derivation of* $\vdash G$ *can be transformed into such normal-form derivation that every cut formula is maximal with respect to such formulas from both the succedent of the first premise and the antecedent of the second premise that are not* G, *its subformulas, or instances of proper subformulas of nonlogical-axiom formulas.*

Consider the following rules:

$$\frac{A, \Gamma \vdash \Pi}{A \circ B, \Gamma \vdash \Pi}\ L2^{+} \qquad \frac{B, \Gamma \vdash \Pi}{A \circ B, \Gamma \vdash \Pi}\ L2^{*} \qquad \frac{\Gamma \vdash A, \Pi}{\Gamma \vdash A \circ B, \Pi}\ R2^{+} \qquad \frac{\Gamma \vdash B, \Pi}{\Gamma \vdash A \circ B, \Pi}\ R2^{*}$$

$$\frac{A, \Gamma \vdash \Pi}{\Gamma \vdash A \circ B, \Pi}\ F2^{+} \qquad \frac{\Gamma \vdash B, \Pi}{\Gamma \vdash A \circ B, \Pi}\ F2^{*} \qquad \frac{A, \Gamma \vdash \Pi}{A \circ B, \Gamma \vdash \Pi}\ B2^{+} \qquad \frac{\Gamma \vdash B, \Pi}{A \circ B, \Gamma \vdash \Pi}\ B2^{*}$$

$$\frac{A, \Gamma \vdash \Pi \quad B, \Delta \vdash \Sigma}{A \circ B, \Gamma \cup \Delta \vdash \Pi \cup \Sigma}\ LA^{*} \qquad \frac{\Gamma \vdash A, \Pi \quad \Delta \vdash B, \Sigma}{\Gamma \cup \Delta \vdash A \circ B, \Pi \cup \Sigma}\ RA^{*}$$

Definition 4. *The calculi obtained from L'_A calculi with weakening by adding the $L2^{+}, R2^{+}, L2^{*}, R2^{*}, F2^{+}, B2^{+}, F2^{*}, B2^{*}$ rules and replacing the LA, RA rules with the LA^{*}, RA^{*} rules, respectively, are called L''_A. The L'_A calculi without weakening have identical L''_A counterparts.*

Proposition 3. *For any L'_A calculus and its L''_A counterpart, any L'_A derivation can be transformed into a L''_A derivation with the same endsequent and vice versa.*

Proposition 4. *For a consistent L'_A calculus with non-reducible nonlogical axioms, every derivation with endsequent $\vdash G$ can be transformed into a normal-form L''_A derivation with the same endsequent and without the weakening rules.*

4. Conclusion

The subformula property is a desirable property for any calculus. This property is a corollary and a primary reason for cut elimination. In general, cut elimination is not possible for sequent calculi with nonlogical axioms. The normal-form theorem shows that derivations can be limited to those satisfying the weak subformula property for a wide class of calculi with nonlogical axioms even though cut is not admissible in them. This theorem gives other constraints for inference rules. Weakening can be embedded into logical rules. Theorem 2 adapts ordered resolution [3] to sequent derivations. It states an additional constraint for the cut rule.

Due to the weak subformula property, the choices for A in the logical axiom $A \vdash A$, the choices for the weakening formulas, and the choices for the principal formulas of logical rules can be limited to the goal, its subformulas, and instances of formulas from nonlogical axioms and their subformulas. Given that the majority of formulas in nonlogical axioms are expected to be shallow, the sets of their subformulas are rather small.

The instantiation of nonlogical axioms, formulas in the logical axiom, and the weakening formulas is a potential source of infinite branching in inference procedures. Fortunately, this problem is solved by using formulas from nonlogical axioms and their subformulas 'as is' after renaming object variables and by embedding unification in inference rules [9]. There exist efficient unification algorithms [8]. They are applicable to quantifier-free first-order formulas because these formulas can be treated as terms whose signature is extended with predicates and logical connectives.

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Alexander Sakharov
Synstretch
Framingham, Massachusetts, USA
e-mail: mail@sakharov.net