A Normal Form of Derivations for Quantifier-Free Sequent Calculi With Nonlogical Axioms

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Logical Characterization of AI Systems

- Logical characterizations give formal descriptions of AI systems and make their results explainable
- AI systems are characterized by a variety of mostly non-standard calculi
- Models may not be available
- The core of AI systems is domain knowledge in the form of logical programs, knowledge base rules, etc. This knowledge is to be axiomatized in the calculi.
- Domain knowledge languages are mostly quantifier-free. Skolem functions may be used in lieu of quantifiers in them.
- Sequent calculi and Hilbert-type calculi
- Sequent calculi have many theoretical advantages, but they do not facilitate inference methods, and the cut rule is not admissible in the presence of axioms representing domain knowledge

Sequent Calculus Framework

Sequent notation for characterizing AI systems. Framework for an assortment of quantifier-free sequent calculi with various logical connectives.

- Formulas are built recursively from atoms and connectives, atom arguments are terms built recursively from object variables, constants, and functions
- Unary and binary logical connectives
- Sequent antecedents and succedents are multisets of formulas
- Standard logical axiom: $A \vdash A$
- Standard structural inference rules (weakening, contraction, cut), or their subset (substructural calculi)
- Logical inference rules with one or two premises
- Domain knowledge is represented by nonlogical axioms in the form of sequents composed of formulas, no metavariables in these axioms
- Inference goals: $\vdash G$

Introduction Rules

Definition. *A nonlogical axiom is called reducible if it has an instance with two or more identical formulas.*

Definition. *A calculus is called consistent if sequent ⊢ is not derivable*

- ◊ - unary connective, ° - binary connective.
- Upper-case Latin letters are formula metavariables, upper-case Greek letters are multiset metavariables

Definition. *A logical inference rule is called an introduction rule if it has one of the forms on the next slide and does not have any applicability provisos.*

The idea of introduction rules is that every formula from a premise is a subformula of some formula from the conclusion, there is a formula in the conclusion that is not identical to any formula from premises. The choice of the introduction rule forms is dictated by the desideratum of the subformula property.

Introduction Rules

$$\frac{A,\Gamma\vdash\Pi}{\circ A,\Gamma\vdash\Pi}\,L1 \qquad \frac{A\vdash\circ\Pi}{\circ A\vdash\circ\Pi}\,LP \qquad \frac{\Gamma\vdash A,\Pi}{\Gamma\vdash\circ A,\Pi}\,R1 \qquad \frac{\circ\Gamma\vdash A}{\circ\Gamma\vdash\circ A}\,RP$$

$$\frac{A,\Gamma\vdash\Pi}{\Gamma\vdash\circ A,\Pi}\,F1 \qquad \frac{\Gamma\vdash A,\Pi}{\circ A,\Gamma\vdash\Pi}\,B1$$

$$\frac{\Gamma\vdash}{\circ\Gamma\vdash}\,LO \qquad \frac{\Gamma\vdash A}{\circ\Gamma\vdash\circ A}\,RL \qquad \frac{\vdash\Pi}{\vdash\circ\Pi}\,RO \qquad \frac{A\vdash\Pi}{\circ A\vdash\circ\Pi}\,LR$$

$$\frac{A,B,\Gamma\vdash\Pi}{A\circ B,\Gamma\vdash\Pi}\,L2 \qquad \frac{\Gamma\vdash A,B,\Pi}{\Gamma\vdash A\circ B,\Pi}\,R2 \qquad \frac{A,\Gamma\vdash B,\Pi}{\Gamma\vdash A\circ B,\Pi}\,F2 \qquad \frac{A,\Gamma\vdash B,\Pi}{A\circ B,\Gamma\vdash\Pi}\,B2$$

$$\frac{A,\Gamma\vdash\Pi \quad B,\Gamma\vdash\Pi}{A\circ B,\Gamma\vdash\Pi}\,LA \qquad \frac{B,\Gamma\vdash\Pi \quad B,\Delta\vdash\Sigma}{A\circ B,\Gamma,\Delta\vdash\Pi,\Sigma}\,LM$$

$$\frac{\Gamma\vdash A,\Pi \quad \Gamma\vdash B,\Pi}{\Gamma\vdash A\circ B,\Pi}\,RA \qquad \frac{\Gamma\vdash A,\Pi \quad \Delta\vdash B,\Sigma}{\Gamma,\Delta\vdash A\circ B,\Pi,\Sigma}\,RM$$

$$\frac{B,\Gamma\vdash\Pi \quad \Delta\vdash A,\Sigma}{\Gamma,\Delta\vdash A\circ B,\Pi,\Sigma}\,FM \qquad \frac{B,\Gamma\vdash\Pi \quad \Delta\vdash A,\Sigma}{A\circ B,\Gamma,\Delta\vdash\Pi,\Sigma}\,BM$$

L_A Calculi

Definition. *A sequent calculus is called a L_A calculus if it has one logical axiom $A \vdash A$ and possibly nonlogical axioms, the cut rule, possibly the two weakening rules, possibly the two contraction rules, some introduction logical rules, and*

- *for every unary connective, the rules with this connective are limited to one R1 rule and possibly one L1 or LP rule, one RP rule and possibly one L1 rule, one F1 rule and possibly one B1 rule, one RL rule and one of LO/L1 rules, or one LR rule and one of RO/R1 rules,*
- *for every binary connective, the rules with this connective are limited to one R2 rule and possibly one LA rule, one R2 rule and possibly one LM rule, one RA rule and possibly one L2 rule, one RM rule and possibly one L2 rule, one F2 rule and possibly one BM rule, or one FM rule and possibly one B2 rule.*

Examples of L_A calculi: quantifier-free fragments of classical and intuitionistic first-order logics, multiplicative linear logic, logic of evaluable non-Horn knowledge bases, modal logic S4, standard deontic logic

L'_A Calculi

Let [Γ] denote the result of applying zero or more possible contractions to multiset Γ. If a calculus set does not include contraction, then [Γ] = Γ. If a calculus includes both weakening and contraction, then the [ ] operation eliminates all duplicate formulas. If a calculus includes contraction and does not include weakening, then this operation is non-deterministic.

Definition. *The calculi obtained from L_A by applying [ ] to both antecedent and succedent in the conclusion of cut and logical inference rules are called L'_A.*

Proposition 1. *For any L_A calculus and its L'_A counterpart, any L_A derivation can be transformed into a L'_A derivation with the same endsequent and vice versa.*

Proposition 2. *The contraction rules are admissible in L'_A derivations for calculi with non-reducible nonlogical axioms.*

Normal Form

Theorem 1. *For a consistent L'_A calculus with non-reducible nonlogical axioms, every derivation with endsequent $\vdash G$ can be transformed into such derivation with the same endsequent and without contractions that the following holds:*

1) *(weak subformula property) Every formula in the derivation is G, its subformula, or an instance of a formula from a nonlogical axiom or its subformula.*
2) *Every cut formula is an instance of a formula from a nonlogical axiom.*
3) *If one premise of cut is the conclusion of a logical rule, then the cut formula is principal in the logical rule and the other premise is a nonlogical axiom or the conclusion of another cut.*
4) *The conclusion of every weakening is the premise of L2, R2, F2, B2, LA, RA, or another weakening.*
5) *Every weakening formula is active in the first descendant L2, R2, F2, B2 rule or adds a formula to the context of a premise of the first descendant LA, RA rule from the context of the other premise of the latter rule.*

The proof is mostly by using standard cut elimination permutations.

Ordered Inference

Definition**.** *Order relation > on formulas is called a simplification order if it is:*

- *well-founded: for any formula s, there is no infinite sequence of formulas s > t > …*
- *monotone: if r is a proper subformula of l, then l > r*
- *stable: if l > r, then $l\theta > r\theta$ for any substitution θ*

Theorem 2. *For a L'_A calculus without LP, RP rules and a simplification order, every derivation of $\vdash G$ can be transformed into such normal-form derivation that every cut formula is maximal with respect to such formulas from both the succedent of the first premise and the antecedent of the second premise that are not G, its subformulas, or instances of proper subformulas of nonlogical-axiom formulas.*

The proof is by using rule permutations.

Embedded Weakening

$$\frac{A, \Gamma \vdash \Pi}{A \circ B, \Gamma \vdash \Pi}\ L2^{+} \qquad \frac{B, \Gamma \vdash \Pi}{A \circ B, \Gamma \vdash \Pi}\ L2^{*} \qquad \frac{\Gamma \vdash A, \Pi}{\Gamma \vdash A \circ B, \Pi}\ R2^{+} \qquad \frac{\Gamma \vdash B, \Pi}{\Gamma \vdash A \circ B, \Pi}\ R2^{*}$$

$$\frac{A, \Gamma \vdash \Pi}{\Gamma \vdash A \circ B, \Pi}\ F2^{+} \qquad \frac{\Gamma \vdash B, \Pi}{\Gamma \vdash A \circ B, \Pi}\ F2^{*} \qquad \frac{A, \Gamma \vdash \Pi}{A \circ B, \Gamma \vdash \Pi}\ B2^{+} \qquad \frac{\Gamma \vdash B, \Pi}{A \circ B, \Gamma \vdash \Pi}\ B2^{*}$$

$$\frac{A, \Gamma \vdash \Pi \quad B, \Delta \vdash \Sigma}{A \circ B, \Gamma \cup \Delta \vdash \Pi \cup \Sigma}\ LA^{*} \qquad \frac{\Gamma \vdash A, \Pi \quad \Delta \vdash B, \Sigma}{\Gamma \cup \Delta \vdash A \circ B, \Pi \cup \Sigma}\ RA^{*}$$

Embedded Weakening

Definition. The calculi obtained from L'_A calculi with weakening by adding the L2+, R2+, L2*, R2*, F2+, B2+, F2*, B2* rules and replacing the LA, RA rules with the LA*,RA* rules, respectively, are called L''_A. The L'_A calculi without weakening have identical L''_A counterparts.

Proposition 3. For any L'_A calculus and its L''_A counterpart, any L'_A derivation can be transformed into a L''_A derivation with the same endsequent and vice versa.

Proposition 4. For a consistent L'_A calculus with non-reducible nonlogical axioms, every derivation with endsequent $\vdash G$ can be transformed into a normal-form L''_A derivation with the same endsequent and without the weakening rules.

Inference

- Derivations can be limited to those satisfying the weak subformula property. This property applies to formulas in the logical axiom, to weakening formulas, and to principal formulas of logical rules.
- Both the normal-form theorem and the ordered inference theorem give constraints for cut application (cut is likely the most frequently used rule when numerous nonlogical axioms are present)
- Due to the normal-form theorem infinite branching can be eliminated by employing the weak subformula property, by merging weakenings with logical rules, and by embedding unification into inference rules

Contributions

- Applying cut elimination techniques to the sequent calculi in which the cut rule is essential and achieving the weak subformula propery without cut elimination.
- Adaptation of ordered resolution to sequent calculi with the cut rule. Simple syntactic proof unlike semantic proofs for ordered resolution.

Discussion

- Concrete introduction rules vs a general form of the rules
- Other logics: non-introduction rules, additional logical axioms, hypersequent calculi
- Research of sequent calculi with nonlogical axioms

Q & A