Some observations on degree 3 and 4 exponential sums over finite fields

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Abstract. By numerical experiments, it is discovered some strictures in distribution of cubic and quartic exponential sums of additive type in finite fields. Concerning the cubic sums, we give a theoretical explanation for that. For the quartic sums, we observe numerically that Euler's deltoid play role in their distribution.

Introduction

Consider the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ of prime order p, its additive character

$$x \mapsto e_p(x) = \exp(2\pi i x/p), \quad x \in \mathbb{F}_p,$$

a one-variable polynomial f over $\mathbb{F}_p$ and an additive type exponential sum

$$S_p(f) = \sum_{x \in \mathbb{F}_p} e_p\big(f(x)\big).$$

The Weil inequality $|\, S_p(f) \,| \le (\deg f - 1)\sqrt{p}$ is valid for all the sums whenever $p \nmid \deg f$. That means, the points

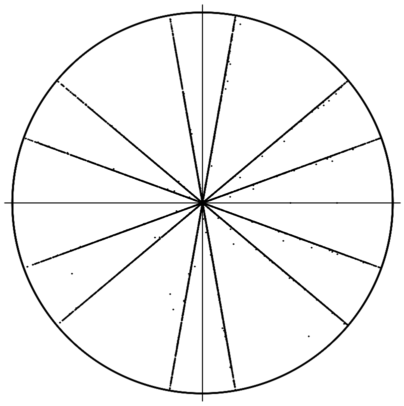
$$E_p(f) = \frac{1}{(\deg f - 1)\sqrt{p}}\, S_p(f)$$

are located in the unit disk $D = \big\{ z \in \mathbb{C} \;\big|\; |z| \le 1 \big\}$. See [1], [2].

Given a one-variable polynomial f over $\mathbb{Z}$, consider f as a polynomial over each of $\mathbb{F}_p$ just by reduction its coefficients $\bmod\, p$. Then one may look on distribution of the points $E_p(f)$ (with prime $p = 2, 3, 5, 7 \ldots$) in the disk D. We have used computer algebra systems PARI and MAPLE to study numerically the sums $S_p(f)$ for lot of polynomials f of degree 3 and 4.

Cubic sums

Consider one instructive sample in [3]. On the picture below we have plotted the real coordinate axis, the imaginary coordinate axis, the unit disk $D \subset \mathbb{C}$, and the points $E_p(f) \in D$ for the polynomial $f(x) = 6x^3 + 3x^2 + 4x$ and for all prime $p \leq 100000$.



The points $E_p(f)$ are concentrated mainly along few lines passing through the point 0. One has a similar picture for other polynomials as well. The number of lines depends on f.

To state our results, let us agree to write $\{t\}$ for the fractional part of $t \in \mathbb{R}$. We have proved [4] the following two propositions.

Consider a cubic polynomial $f(x) = ax^3 + bx^2 + cx$ *over* $\mathbb{Z}$. *Let* l *be an integer,* $\gcd(l, 3a) = 1$, *and let* p *be any prime under the conditions* $lp + 1 \equiv 0 \bmod 27a^3$ *and* $p \nmid 6a$. *If* $S_p(f) \neq 0$, *then the real axis forms the angle*

$$\theta_p = 2\pi \left\{ \frac{b\,(2b^2 - 9ac)}{27a^2} \Big(l + \frac{1}{p} \Big) \right\}$$

with the line passing through the points 0 *and* $S_p(f)$.

This proposition implies easily the second one.

Consider a cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$ *over* $\mathbb{Z}$. *The points* $E_p(f)$ *are concentrated along the lines that pass through the point* 0 *and intersect the real axis under the angles*

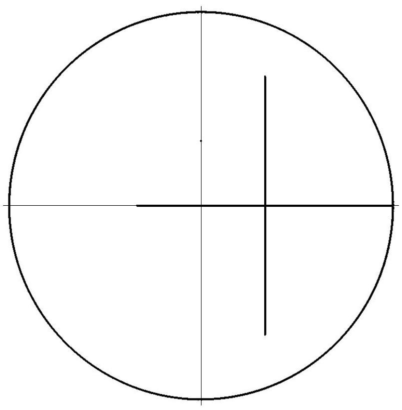
$$\theta = 2\pi \left\{ \frac{b\,(2b^2 - 9ac)}{27a^2}\, l \right\}$$

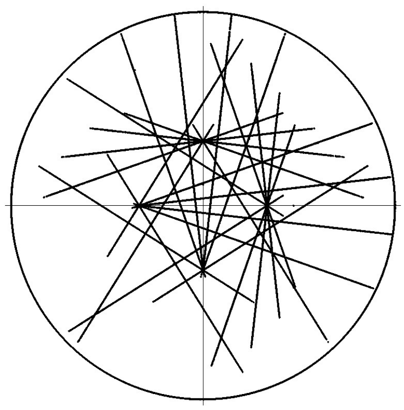
with $l \in \mathbb{Z}$ *under the condition* $\gcd(l, 3a) = 1$.

This result gives us full description of the asters attached to cubic polynomials in [3]. Also, it shows that there are no at all the clusters considered in [3].

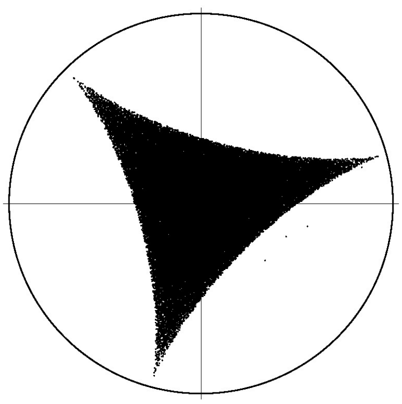
Quartic sums

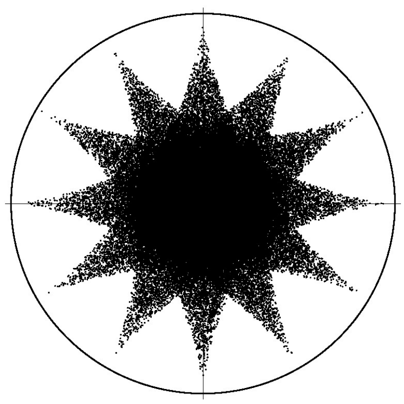
For some quartic polynomials f, we have find empirically that almost all of the points $E_p(f)$ are located on few intervals in D. Let us look on two samples. On the pictures below we have plotted the real and imaginary coordinate axes, the disk $D \subset \mathbb{C}$, and the points $E_p(f) \in D$ for chosen polynomials f and for all prime $p \leq 480000$. The sums $S_p(f)$ with $f(x) = x^4$ are nothing but the biquadratic Gauss sums. By known explicit formulas, one has either $E_p(f) = i/3$ or $E_p(f) \in [-1/3, 1]$ or $E_p(f) \in [1/3 - 2i/3, 1/3 + 2i/3]$ according to $p \equiv 3 \pmod 4$ or $p \equiv 1 \pmod 8$ or $p \equiv 5 \pmod 8$. This case is represented by the left-hand side picture below.





The right-hand side picture represents similarly the case $f(x) = 7x^4 + x^2$. Assume $r \in \mathbb{Z}$ and $\gcd(r, 56) = 1$. It seems reasonable to expect that the points $E_p(f)$ with $p \equiv r \pmod{56}$ form one of 24 intervals shown on the picture.
There are other polynomials f with entirely different distribution of the points $E_p(f)$. Two typical samples are given on the pictures below.

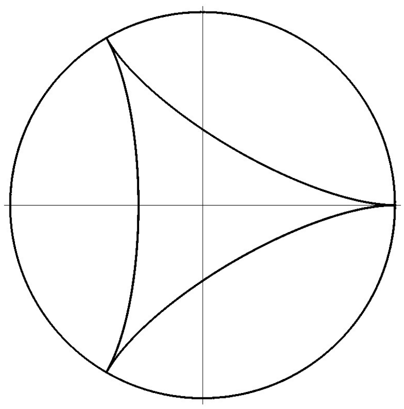




For the polynomial $f(x) = 4x^4 + 8x^3 + 3x^2 + 6$, the points $E_p(f)$ with $p \leq 1000000$ forms the left-hand side picture. We see that the points $E_p(f)$ are located within some three-cusped curve. The right-hand side picture is formed similarly for $f(x) = 7x^4 + 1$. For a lot of polynomials f, we have similar pictures "formed by 1, 2, 4, 8 triangles bounded by the same three-cusped curve".

We conjecture that the three-cusped curve discussed is the Euler deltoid considered in 1745 *in connection with an optical problem.*

The deltoid can be defined as the curve consisting of the points $z = x + iy \in \mathbb{C}$ satisfying $3(x^2 + y^2)(x^2 + y^2 + 2) = 8x^3 - 24xy^2 + 1$ with $x, y \in \mathbb{R}$.



Also, the deltoid can be created by a point on the circumference of a circle of radius 1/3 as it rolls without slipping along the inside of a circle of radius 1.

References

[1] J.-P. Serre, *Majorations de sommes exponentielles*, Société Mathématique de France, Asterisque 41–42, p. 111–126, 1977.

[2] S. A. Stepanov, *Arithmetic of algebraic curves*, Moscow, 1991 (in Russian). English translation: Springer–Verlag, 1995.

[3] N. V. Proskurin, *On some cubic exponential sums*, Zap. Nauchn. semin. POMI, vol. 502, 122–132, 2021 (in Russian).

[4] N. V. Proskurin, *Distribution of cubic exponential sums*, Zap. Nauchn. semin. POMI, vol. 511, 161–170, 2022 (in Russian).

[5] N. V. Proskurin, *On quartic exponential sums*, Zap. Nauchn. semin. POMI, vol. 517, 162–175, 2022 (in Russian).

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